A Refinement of an Inequality for Positive Operators on Pseudo-Hilbert Spaces

Loredana Ciurdariu
Department of Mathematics, ”Politehnica” University of Timisoara
P-ta. Victoriei, No. 2, 300006-Timișoara, Romania

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Abstract

In this paper, several inequalities for positive definite operators defined on pseudo-Hilbert spaces and Hilbert spaces respectively will be presented under suitable assumptions, starting from some refinements of the Kittaneh-Manasrah inequality which improves the well-known inequality of Young. The results will improve the inequalities presented before for positive operators in Hilbert spaces.

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1. Introduction

In this paper, it is necessary to recall the following results which are given in the papers [4] and [5] and will be used below in the demonstration of our results. In demonstrations of the main section the same method as in the paper [3] will be utilized.

Lemma 1. ([4]) Let \(a\) and \(b\) be such that \(a, b \geq 0\) and \(0 \leq \nu \leq 1\). Then the following inequality holds:

\[ \nu a^2 + (1 - \nu)b^2 \leq (a^\nu b^{1-\nu})^2 + s_0(a-b)^2, \]

where \(s_0 = \max\{\nu, 1 - \nu\} \).
Lemma 2. ([5]) For all \( x, y \) positive real numbers and \( \lambda \in (0, 1) \) we have the inequality

\[
2rE\left(x, y, \frac{1}{2}\right) \leq E(x, y, \lambda) \leq 2(1-r)E\left(x, y, \frac{1}{2}\right),
\]

where

\[
E(x, y, \lambda) = \lambda \exp x + (1-\lambda) \exp y - \exp (\lambda x + (1-\lambda)y) - \frac{\lambda(1-\lambda)}{2}(x-y)^2
\]

and \( r = \min\{\lambda, 1-\lambda\} \).

Theorem 1. ([5]) For \( a, b \geq 1, \) and \( \lambda \in (0, 1) \) we have

\[
r(\sqrt{a} - \sqrt{b})^2 + A_1(\lambda) \log^2 \left( \frac{a}{b} \right) \leq \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} \leq (1-r)(\sqrt{a} - \sqrt{b})^2 + B_1(\lambda) \log^2 \left( \frac{a}{b} \right)
\]

where \( r = \min\{\lambda, 1-\lambda\} \), \( A_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4} \) and \( B_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4} \).

First, it is necessary to recall that for selfadjoint operators \( A, B \in B(H) \) we write \( A \leq B \) (or \( B \geq A \)) if \( \langle Ax, x \rangle \leq \langle Bx, x \rangle \) for every vector \( x \in H \). We will consider for beginning \( A \) as being a selfadjoint linear operator on a complex Hilbert space \( \langle H; \langle, \rangle \rangle \) as in [3] and the references therein. The Gelfand map establishes a \(*\)-isometrically isomorphism \( \Phi \) between the set \( C(Sp(A)) \) of all \textit{continuous functions} defined on the spectrum of \( A \), denoted \( Sp(A) \), and the \( C^*\)- algebra \( C^*(A) \) generated by \( A \) and the identity operator \( 1_H \) on \( H \) as follows: For any \( f, f \in C(Sp(A)) \) and for any \( \alpha, \beta \in \mathbb{C} \) we have

(i) \( \Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g) \);
(ii) \( \Phi(fg) = \Phi(f)\Phi(g) \) and \( \Phi(f) = \Phi(f^*) \);
(iii) \( \|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)| \);
(iv) \( \Phi(f_0) = 1_H \) and \( \Phi(f_1) = A \), where \( f_0(t) = 1 \) and \( f_1(t) = t \) for \( t \in Sp(A) \).

Using this notation, as in [3] for example, we define

\[
f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A))
\]

and we call it the \textit{continuous functional calculus} for a selfadjoint operator \( A \). It is known that if \( A \) is a selfadjoint operator and \( f \) is a real valued continuous function on \( Sp(A) \), then \( f(t) \geq 0 \) for any \( t \in Sp(A) \) implies that \( f(A) \geq 0 \), i.e. \( f(A) \) is a \textit{positive operator} on \( H \). In addition, if and \( f \) and \( g \) are real valued functions on \( Sp(A) \) then the following property holds:

(1) \( f(t) \geq g(t) \) for any \( t \in Sp(A) \) implies that \( f(A) \geq g(A) \)
in the operator order of \( B(H) \).
We recall the definitions of pseudo-Hilbert spaces (so called Lonyes $Z$-spaces) and of the admissible spaces in the Lonyes sense and then as in [2] we use the functional calculus with functions of the class $C_1$ in results which will be proved below.

A locally convex space $Z$ is called admissible in the Lonyes sense if the following conditions are satisfied:

(A.1) $Z$ is complete;
(A.2) there is a closed convex cone in $Z$, denoted $Z_+$, defines an order relation on $Z$ (that is $z_1 \leq z_2$ if $z_2 - z_1 \in Z_+$);
(A.3) there is an involution in $Z$, $z \mapsto z^*$ (that is $z^{**} = z$, $(\alpha z)^* = \overline{\alpha} z^*$, $(z_1 + z_2)^* = z_1^* + z_2^*$) such that $z \in Z_+$ implies $z^* = z$;
(A.4) the topology of $Z$ is compatible with the order (that is there exists a basis of convex solid neighbourhoods of the origin);
(A.6) any monotonously decreasing sequence in $Z_+$ is convergent.

Let $Z$ be an admissible space in the Lonyes sense. A topological linear space $\mathcal{H}$ is called pre-Lonyes $Z$-space if it satisfies the following properties:

(L1) $\mathcal{H}$ is endowed with an $Z$-valued inner product (gramian), i.e. there exists an application $(h, k) \in \mathcal{H} \times \mathcal{H} \to [h, k] \in Z$ having the properties:

(G.1) $[h, h] \geq 0$; $[h, h] = 0$ implies $h = 0$;
(G.2) $[h_1 + h_2, h] = [h_1, h] + [h_2, h]$;
(G.3) $[\lambda h, k] = \lambda [h, k]$;
(G.4) $[h, k]^* = [k, h]$;

for all $h, k, h_1, h_2 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

(L.2) The topology of $\mathcal{H}$ is the weakest locally convex topology on $\mathcal{H}$ for which the application $h \in \mathcal{H} \to [h, h] \in Z$ is continous.

Moreover, if $\mathcal{H}$ is a complete spaces with this topology, then $\mathcal{H}$ is called Lonyes $Z$-space (pseudo-Hilbert space).

Let $A \in \mathcal{B}_h^*(\mathcal{H})$, where $\mathcal{H}$ is now a pseudo-Hilbert space (so called Lonyes $Z$-spaces) and

$$m_A := \sup \{ \mu : \mu [h, h] \leq [Ah, h], h \in \mathcal{H} \},$$

$$M_A := \inf \{ \nu : [Ah, h] \leq \nu [h, h], h \in \mathcal{H} \}.$$

We say that the function $f$ is in the class $C_1$ and we denote $f \in C_1 [m_A, M_A]$ if $f$ is positive and superior semicontinuous on $[m_A, M_A]$.

We will denote by $\overline{A(A)}^t$ the strong closing of $A(A)$ in $\mathcal{B}^* (\mathcal{H})$.

The mapping

$$f : C_1 [m_A, M_A] \to \overline{A(A)}^t, \quad f \to f(A)$$

by which to a function $f \in C_1 [m_A, M_A]$ we associate the gramian self-adjoint operator denoted by $f(A)$ and defined by $f(A) = \lim_{n \to \infty} p_n(A)$ where $p_n$ is a decreasing sequence of polynomials $p_n$ with $p(\lambda) = \lim_{n \to \infty} p_n(\lambda)$ for any $\lambda \in [m_A, M_A]$, is called functionals calculus with functions in class $C_1$.

**Theorem 2.** (Lemma 2.1.1[2]) The functional calculus with functions of the class $C_1$ has the following immediate properties:

(i) the mapping $f \to f(A)$ is monotone;
(ii) \( f \to f(A) \) is function of positive type and positively homogeneous;
(iii) \( f \to f(A) \) is additive and multiplicative (all these three properties being inherited by passing to the limit from the functional calculus with polynomials defined at the beginning);
(iv) In addition, the functional calculus with functions of the class \( C_1 \) extends the functional calculus with continuous and positive functions on \( \sigma(A) \) defined in Corollary 1.5.6,
\[
f : C_+(\sigma(A)) \to \mathcal{A}(A), \quad f \to f(A)
\]
if \( A \in \mathcal{B}_h^\ast(\mathcal{H}) \).

Using the definition from [3], we say that the functions \( f, g : [a, b] \to \mathbb{R} \) are synchronous (asynchronous) on the interval \( [a, b] \) if they satisfy the following condition:
\[
(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0
\]
for each \( t, s \in [a, b] \).

2. Main results

The following results present several inequalities for functions of positive operators when we have a classic inner product which takes its values in \( \mathbb{C} \) and the second inner product (gramian) takes its values in an admissible space \( Z \).

**Proposition 1.** Let \( A \) be a positive definite operator on the Hilbert space \( H \), \( A \in \mathcal{B}(H) \) and \( B \) a positive operator on the pseudo-Hilbert space \( K \), \( B \in \mathcal{B}_h^\ast(K) \). Then we have
\[
\nu < A^2x, x > [y, y] + (1 - \nu) < x, x > [B^2y, y] \leq
\]
\[
< A^{2\nu}x, x > [B^{2(1-\nu)}y, y] + s_0 (< A^2x, x > [y, y] - 2 < Ax, x > [By, y] + < x, x > [B^2y, y])
\]
for each \( x \in H \) and \( y \in K \), where \( 0 \leq \nu \leq 1 \) and \( s_0 = \max\{\nu, 1 - \nu\} \).

**Proof.** We consider the continue function \( f(a) = (a^\nu b^{1-\nu})^2 + s_0(a - b)^2 - (\nu a^2 + (1 - \nu)b^2) \), which is positive for \( a \geq 0 \) and we fix \( b \geq 0 \) and then by the property (1) for each \( x \in H \) we have that
\[
< (\nu A^2 + (1 - \nu)b^2I)x, x > \leq < [A^{2\nu}b^{2(1-\nu)} + s_0(A^2 - 2Ab + b^2I)]x, x >
\]
which is equivalent with
\[
\nu < A^2x, x > +(1 - \nu)b^2 < x, x > \leq
\]
\[
b^{2(1-\nu)} < A^{2\nu}x, x > + s_0<[A^2x, x] > -2b < Ax, x > + b^2 < x, x >
\]
for each \( b > 0 \).

If we apply now Theorem 2 for last inequality taking into account that
\[
\nu < A^2x, x > +(1 - \nu)b^2 < x, x > \geq 0,
\]
then for any \( y \in K \) we get
\[
[(\nu < A^2x, x > +(1 - \nu) < x, x > B^2)y, y] \leq
\]
\[
< (B^{2(1-\nu)} < A^{2\nu}x, x > + s_0(< A^2x, x > I_K - 2B < Ax, x > + B^2 < x, x >))y, y >
\]
and this inequality is equivalent with
\[ \nu < A^2 x, x > [y, y] + (1 - \nu) < x, x > [B^2 y, y] \leq \]
\[ \leq < A^{2\nu} x, x > [B^{2(1-\nu)} y, y] + s_0 ( < A^2 x, x > [y, y] - 2 < A x, x > [B y, y] + < x, x > [B^2 y, y] ) \]
for each \( x \in H \) and \( y \in \mathcal{K} \).

\[ \square \]

**Theorem 3.** Let \( A \) be a positive definite operator on the Hilbert space \( H \), \( A \in B(H) \) and \( B \) a positive operator on the pseudo-Hilbert space \( \mathcal{K} \), \( B \in B^*_1(\mathcal{K}) \). Then the following inequality holds:

\[
\begin{align*}
& r \{ < \exp(A) x, x > [y, y] + < x, x > [\exp(B) y, y] - 2 < \exp\left(\frac{A}{2}\right) x, x > [\exp\left(\frac{B}{2}\right) y, y] - \frac{1}{4} ( < A^2 x, x > [y, y] - 2 < A x, x > [B y, y] + < x, x > [B^2 y, y] ) \} \leq \\
& \leq \lambda < \exp(A) x, x > [y, y] + (1 - \lambda) < x, x > [\exp(B) y, y] - < \exp(\lambda A) x, x > \\
& \cdot [\exp((1-\lambda)B) y, y] - \frac{\lambda(1-\lambda)}{2} ( < A^2 x, x > [y, y] - 2 < A x, x > [B y, y] + < x, x > [B^2 y, y] ) \leq \\
& \leq (1 - r) \{ < \exp(A) x, x > [y, y] + < x, x > [\exp(B) y, y] - 2 < \exp\left(\frac{A}{2}\right) x, x > [\exp\left(\frac{B}{2}\right) y, y] - \\
& - \frac{1}{4} ( < A^2 x, x > [y, y] - 2 < A x, x > [B y, y] + < x, x > [B^2 y, y] ) \}. \\
\end{align*}
\]

for each \( x \in H \) and \( y \in \mathcal{K} \), where \( r = \min\{\lambda, 1 - \lambda\} \).

**Proof.** We write and then use the inequality from Lemma 2 with \( x \) replaced by \( a \) and \( y \) replaced by \( b \) obtaining:

\[
r \left[ \exp(a) + \exp(b) - 2 \exp\left(\frac{a+b}{2}\right) - \frac{1}{4}(a-b)^2 \right] \leq \\
\leq \lambda \exp(a) + (1 - \lambda) \exp(b) - \exp(\lambda a + (1 - \lambda) b) - \frac{\lambda(1-\lambda)}{2} (a-b)^2 \leq \\
\leq (1 - r) \left[ \exp(a) + \exp(b) - 2 \exp\left(\frac{a+b}{2}\right) - \frac{1}{4}(a-b)^2 \right].
\]

We fix \( b > 0 \) and apply the property (1) for previous inequality obtaining:

\[
\begin{align*}
& < r [\exp(A) + \exp(b) 1_H - 2 \exp\left(\frac{b}{2}\right) \exp\left(\frac{A}{2}\right) - \frac{1}{4}(A^2 - 2bA + b^2 1_H)] x, x > \leq \\
& \leq [\lambda \exp(A) + (1 - \lambda) \exp(b) 1_H - \exp(\lambda A) \exp((1-\lambda) b) - \frac{\lambda(1-\lambda)}{2} (A^2 - 2bA + b^2 1_H)] x, x > \\
& \leq (1 - r) [\exp(A) + \exp(b) 1_H - 2 \exp\left(\frac{b}{2}\right) \exp\left(\frac{A}{2}\right) - \frac{1}{4}(A^2 - 2bA + b^2 1_H)] x, x >
\end{align*}
\]

which is equivalent with the following

\[ r [\exp(A) x, x > + \exp(b) < x, x > - 2 \exp\left(\frac{b}{2}\right) < \exp\left(\frac{A}{2}\right) x, x > - \]

\[ \]
\[-\frac{1}{4}(< A^2 x, x > -2b < Ax, x > +b^2 < x, x >)] \leq \\
\leq \lambda < \exp(A)x, x > + (1 - \lambda) \exp(b) < x, x > - \exp((1 - \lambda)b) < \exp(\lambda A)x, x > - \\
- \frac{\lambda(1 - \lambda)}{2} (< A^2 x, x > -2b < Ax, x > +b^2 < x, x >) \leq \\
\leq (1 - r)[< \exp(A)x, x > + \exp(b) < x, x > -2 \exp(\frac{b}{2}) < \exp(\frac{A}{2})x, x > - \\
- \frac{1}{4}(< A^2 x, x > -2b < Ax, x > +b^2 < x, x >)], \\
for any \ x \in H.

If we apply Theorem 2 for previous inequality for the variable \( b \), then we have for any \( y \in K \) that

\[
r\{ < \exp(A)x, x > [y, y] + < x, x > [\exp(B)y, y] - 2 < \exp(\frac{A}{2})x, x > [\exp(\frac{B}{2})y, y] - \\
- \frac{1}{4}(< A^2 x, x > [y, y] - 2 < Ax, x > [By, y] + < x, x > [B^2 y, y]) \leq \\
\leq \lambda < \exp(A)x, x > [y, y] + (1 - \lambda) < x, x > [\exp(B)y, y] - < \exp(\lambda A)x, x > [\exp((1 - \lambda)B)y, y] - \\
- \frac{\lambda(1 - \lambda)}{2} (< A^2 x, x > [y, y] - 2 < Ax, x > [By, y] + < x, x > [B^2 y, y]) \leq \\
\leq (1 - r)\{ < \exp(A)x, x > [y, y] + < x, x > [\exp(B)y, y] -2 < \exp(\frac{A}{2})x, x > [\exp(\frac{B}{2})y, y] - \\
- \frac{1}{4}(< A^2 x, x > [y, y] - 2 < Ax, x > [By, y] + < x, x > [B^2 y, y])\}
\]

\[\square\]

**Proposition 2.** Let \( A \) be a positive definite operators on the Hilbert space \( H \), \( A \in B(H) \) and \( B \) a positive operator on the pseudo-Hilbert space \( K \), \( B \in B^+_h(K) \). If \( \text{Sp}(A), \text{Sp}(B) \subseteq [1, \infty) \), and \( \lambda \in (0, 1) \) then we have

\[
r\left( < Ax, x > [y, y] + < x, x > [By, y] - 2 < A^{\frac{1}{2}}x, x > [B^{\frac{1}{2}}y, y] \right) + \\
+ A_1(\lambda) \left[ < (\log^2 A)x, x > [y, y] + < x, x > [(\log^2 B)y, y] - 2 < (\log A)x, x > [(\log B)y, y] \right] \leq \\
\leq \lambda < Ax, x > [y, y] + (1 - \lambda) < x, x > [By, y] - < A^{\lambda}x, x > [B^{1-\lambda}y, y] \leq \\
\leq (1 - r)\left( < Ax, x > [y, y] + < x, x > [By, y] - 2 < A^{\frac{1}{2}}x, x > [B^{\frac{1}{2}}y, y] \right) + \\
+ B_1(\lambda) \left[ < (\log^2 A)x, x > y + < x, x > [(\log^2 B)y, y] - 2 < (\log A)x, x > [(\log B)y, y] \right]
\]

for each \( x \in H \) and \( y \in K \) where \( r = \min\{\lambda, 1 - \lambda\} \), \( A_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4} \) and \( B_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4} \).
Proof. We consider the continue and positive functions \( f(a) = \lambda a + (1 - \lambda) b - a^{1 - \lambda} - r(a + b - 2a^{1/2}b^{1/2}) - A_1(\lambda)[\log^2 a + \log^2 b - 2 \log a \log b] \), and \( g(a) = (1 - r)(a + b - 2a^{1/2}b^{1/2}) + B_1(\lambda)[\log^2 a + \log^2 b - 2 \log a \log b] - \lambda a - (1 - \lambda)b + a^{1 - \lambda} \) which are positive for \( a \geq 1 \) and we fix \( b \geq 1 \) and then by the property (1) for each \( x \in H \) we have that

\[
r(\langle Ax, x \rangle + b < x, x > - 2b^{1/2} < A^{1/2}x, x \rangle) + \]

\[
+ A_1(\lambda) \left[ \langle \log^2 A \rangle x, x \rangle + \log^2 b < x, x > - 2 \log b < (\log A)x, x \rangle \right] \leq
\]

\[
\leq \lambda < Ax, x > + (1 - \lambda) b < x, x > - b^{1 - \lambda} < A^{\lambda}x, x > \leq
\]

\[
\leq (1 - r) \left( \langle Ax, x \rangle + b < x, x > - 2b^{1/2} < A^{1/2}x, x \rangle \right) +
\]

\[
+ B_1(\lambda) \left[ \langle \log^2 A \rangle x, x \rangle + \log^2 b < x, x > - 2 \log b < (\log A)x, x \rangle \right]
\]

for each \( b > 1 \).

If we apply now Theorem 2 for last inequality, then for any \( y \in K \) we get

\[
r \left( \langle Ax, x \rangle + b < y, y > + < x, x > [By, y] - 2 < A^{1/2}x, x > [B^{1/2}y, y] \right) +
\]

\[
+ A_1(\lambda) \left[ \langle \log^2 A \rangle x, x \rangle + < x, x > [(\log^2 B)y, y] - 2 < \langle \log A \rangle x, x > [(\log B)y, y] \right] \leq
\]

\[
\leq \lambda < Ax, x > [y, y] + (1 - \lambda) < x, x > [By, y] - < A^{\lambda}x, x > [B^{1-\lambda}y, y] \leq
\]

\[
\leq (1 - r) \left( \langle Ax, x \rangle + y, y > + < x, x > [By, y] - 2 < A^{1/2}x, x > [B^{1/2}y, y] \right) +
\]

\[
+ B_1(\lambda) \left[ \langle \log^2 A \rangle x, x \rangle + < x, x > [(\log^2 B)y, y] - 2 < \langle \log A \rangle x, x > [(\log B)y, y] \right]
\]

for each \( x \in H \) and \( y \in K \).

Next particular case of Proposition 2 may be of interest as well:

Remark 1. Under previous conditions, if we consider \( K \) as being the Hilbert space \( H, y = x \) and \( A = B \) then the above inequality becomes:

\[
2r \left[ \langle Ax, x > - \left( < A^{1/2}x, x > \right)^2 \right] + 2A_1(\lambda) \left[ \langle \log^2 A \rangle x, x > - \left( < \log A \rangle x, x > \right)^2 \right] \leq
\]

\[
\leq 1 - < A^{1 - \lambda}x, x >
\]

\[
2(1 - r) \left[ \langle Ax, x > - \left( < A^{1/2}x, x > \right)^2 \right] + 2B_1(\lambda) \left[ \langle \log^2 A \rangle x, x > - \left( < \log A \rangle x, x > \right)^2 \right]
\]

In the following, we can think to rewrite some results as Theorem 1, see [3], in the case when we have an inner product which take its values in \( C \) and the second takes values in an admissible space \( Z \).
Theorem 4. Let $A$ be a selfadjoint operator on the Hilbert space $H$, $A \in B(H)$ and $B$ a selfadjoint operator on the pseudo-Hilbert space $K$, $B \in B^*(K)$ with $\text{Sp}(A), \text{Sp}(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \to \mathbb{R}$ are continuous, positive and synchronous (asynchronous) on $[m, M]$, then

$$<f(A)g(A)x, x > [y, y] + <x, x > [f(B)g(B)y, y] \geq (\leq)$$

$$<f(A)x, x > [g(B)y, y] + <g(A)x, x > [f(B)y, y],$$

for any $x \in H$ and $y \in K$.

Proof. We will use the same method as in [3] and we take into account only the case of synchronous functions. Therefore, if we fix $s \in [m, M]$ and apply (1) for inequality

$$f(t)g(t) + f(s)g(s) \geq f(t)g(s) + f(s)g(t)$$

where $t, s \in [m, M]$ then we have for each $x \in H$ that

$$<f(A)g(A)x, x > + f(s)g(s) < x, x > \geq g(s) < f(A)x, x > + f(s) < g(A)x, x >$$

for each $s \in [m, M]$.

Now if we apply Theorem 2 for previous inequality

$$<f(A)g(A)x, x > I_K + f(B)g(B) < x, x > \geq g(B) < f(A)x, x > + f(B) < g(A)x, x >,$$

and then we have for any $y \in K$ that

$$[(<f(A)g(A)x, x > I_K + f(B)g(B) < x, x >)y, y] \geq$$

$$\geq [(g(B) < f(A)x, x > + f(B) < g(A)x, x >)y, y]$$

or by calculus

$$<f(A)g(A)x, x > [y, y] + <x, x > [f(B)g(B)y, y] \geq$$

$$\geq <f(A)x, x > [g(B)y, y] + <g(A)x, x > [f(B)y, y].$$

□

Remark 2. Under previous conditions, if we consider $K$ as being the Hilbert space $H$, $y = x$ and $A = B$ then the above inequality becomes:

$$<x, x > < f(A)g(A)x, x > \geq (\leq) < f(A)x, x > < g(A)x, x >.$$

Remark 3. Under previous conditions, if we consider $K$ as being the Hilbert space $H$, $y = x$ and $A = B$ then the above inequality becomes:

$$<f(A)g(A)x, x > \geq (\leq) < f(A)x, x > < g(A)x, x >,$$

for any $x \in H$ with $||x|| = 1$. 

Theorem 5. Let $A$ be a selfadjoint operator on the Hilbert space $H$, $A \in B(H)$ and $B$ a selfadjoint operator on the pseudo-Hilbert space $K$, $B \in B^*(K)$ with $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \to \mathbb{R}_+$ are continuous, positive and synchronous on $[m, M]$, then

$$[f(B)g(B)y, y] + f(< Ax, x >)g(< Ax, x >)[y, y] \geq f(< Ax, x >)[g(B)y, y] + g(< Ax, x >)[f(B)y, y]$$

for any $x \in H$ and $y \in K$.

Proof. As in [3], using the hypothesis that $f, g$ are synchronous and $m \leq < Ax, x > \leq M$ for any $x \in H$ with $||x|| = 1$ we have

$$(f(t) - f(< Ax, x >)) (g(t) - g(< Ax, x >)) \geq 0$$

for any $t \in [a, b]$ or by calculus

$$f(t)g(t) + f(< Ax, x >)g(< Ax, x >) \geq g(t)f(< Ax, x >) + f(t)g(< Ax, x >).$$

By functional calculus, Theorem 2 we find:

$$f(B)g(B) + f(< Ax, x >)g(< Ax, x >)I_K \geq g(B)f(< Ax, x >) + f(B)g(< Ax, x >)$$

and from here,

$$[f(B)g(B)y, y] + f(< Ax, x >)[g(B)y, y] \geq [f(< Ax, x >)[g(B)y, y] + g(< Ax, x >)[f(B)y, y],$$

for any $y \in K$.

□

References


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