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Clique Cover of the Join and the Corona of Graphs

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Abstract

Let G be a graph. A clique of a graph G is a nonempty subset S of $V(G)$ such that S induces a complete subgraph of G. A family $\mathcal{F}$ of cliques of G is a clique cover of G if for every $v \in V(G)$ there exists $S \in \mathcal{F}$ such that $v \in S$. The clique covering number of G is the minimum cardinality among the clique covers of G. In this paper, we give explicit forms of the clique covering of the join and the corona of graphs in terms of the clique covering numbers of graphs in consideration.

Mathematics Subject Classification: 05C30

Keywords: clique, clique cover, clique covering number

1 Introduction

Let G be a graph. A *clique* of a graph G is a nonempty subset S of $V(G)$ such that S induces a complete subgraph of G. A family $\mathcal{F}$ of cliques of G is a *clique cover* of G if for every $v \in V(G)$ there exists $S \in \mathcal{F}$ such that $v \in S$. The *clique covering number* of G, denoted by $cc(G)$, is given by

$$cc(G) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a clique cover } G\}.$$

2 Join of Two Graphs

Here, we formally define the join of two vertex disjoint graphs.

Definition 2.1 [4] Let G and H be vertex disjoint graphs. The *join* $G+H$ of G and H has vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set

$$E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$$

Consequently,

$$|V(G+H)| = |V(G)| + |V(H)|,$$

and

$$|E(G+H)| = |E(G)| + |E(H)| + |V(G)||V(H)|.$$

Note also that the operation $+$ is commutative, that is, $G+H$ is isomorphic to $H+G$ with respect to adjacency.

Let us consider an illustration of the above definition.

Example 2.2 Consider the path P_9 and the complete graph K_1 with $V(K_1) = \{u\}$. Then the join of P_9 and K_1 is shown below.

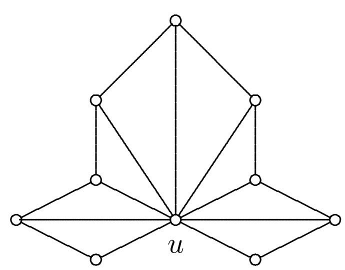

Figure 2.1: The join $P_9 + K_1$.

In the above illustration, every vertex of P_9 is joined with the vertex u and vice versa.

We give the exact value of the clique covering number of the join of graphs $G+H$ in terms of the clique covering numbers of graphs G and H.

Theorem 2.3 *Let G and H be graphs. Then*

$$cc(G+H) = \max\{cc(G), cc(H)\}.$$

Proof: Assume without loss of generality that $cc(G) \leq cc(H)$. Let $\mathcal{F}_G = \{S_1, S_2, \ldots, S_k\}$ be a clique cover of G such that $cc(G) = |\mathcal{F}_G|$. Similarly, let $\mathcal{F}_H = \{S_1{}', S_2{}', \ldots, S_r{}'\}$ be a clique cover of H such that $cc(H) = |\mathcal{F}_H|$. Then $k \leq r$. For each $i = 1, 2, \ldots, k$, let $S_i^* = S_i \bigcup S_i'$. Let $\mathcal{F}_{G+H} = \{S_i^* : i = 1, 2, \ldots, k\} \cup \{S_k{}', \ldots, S_r{}'\}$. Then $\mathcal{F}_{G+H}$ is a clique cover of $G + H$. Hence, $cc(G+H) \leq r$. Suppose that $cc(G+H) < r$. Then, at least one of the cliques in the cover of $G + H$ contains the union $H_i \cup \{v : v \in V(H)\}$. This is a contradiction. Accordingly, $cc(G + H) = r = \max\{cc(G), cc(H)\}$. □

The clique covering number of the generalized wheel is established in the following theorem.

Theorem 2.4 *Let $W_{m,n}$ be the generalized wheel of order $m + n$. Then* $cc(W_{m,n}) = \max\left\{m, \left\lceil \frac{n}{2} \right\rceil\right\}$ *for $n \geq 2$.*

Proof: Note that $W_{m,n} = \overline{K_m} + C_n$. Hence,

$$cc(W_{m,n}) = cc(\overline{K_m} + C_n) = \max\{cc(\overline{K_m}), cc(C_n)\}.$$

Note that, $cc(\overline{K_m}) = m$ and $cc(C_n) = \left\lceil \frac{n}{2} \right\rceil$. Thus, $cc(W_{m,n}) = \max\left\{m, \left\lceil \frac{n}{2} \right\rceil\right\}$.

Corollary 2.5 $cc(W_n) = \left\lceil \frac{n}{2} \right\rceil$ *for $n \geq 4$.*

The clique covering number of the generalized fan is established in the following result.

Theorem 2.6 *Let $F_{m,n}$ be the generalized fan of order $m + n$. Then for* $cc(F_{m,n}) = \max\left\{m, \left\lceil \frac{n}{2} \right\rceil\right\}$.

Proof: Note that $F_{m,n} = \overline{K_m} + P_n$. Hence,

$$cc(F_{m,n}) = cc(\overline{K_m} + P_n) = \max\{cc(\overline{K_m}), cc(P_n)\}.$$

Since $cc(\overline{K_m}) = m$ and $cc(P_n) = \left\lceil \frac{n}{2} \right\rceil$, we have $cc(F_{m,n}) = \max\left\{m, \left\lceil \frac{n}{2} \right\rceil\right\}$. □

Corollary 2.7 $cc(F_n) = \left\lceil \frac{n}{2} \right\rceil$ *for $n \geq 4$.*

3 Corona of Two Graphs

Formally, we define the corona of two graphs.

Definition 3.1 [4] The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.

Example 3.2 The figure below illustrates the corona $P_3 \circ C_3$.

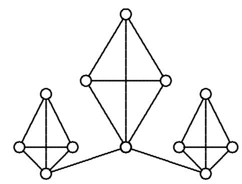

Figure 3.1: The corona $P_3 \circ C_3$.

Theorem 3.3 *Let G and H be connected graphs. Then*

$$cc(G \circ H) = cc(H)|V(G)|.$$

Proof: For each vertex $v \in V(G)$, let $G_1 = \bigcup_{v \in V(G)} H_v$, where H_v is a copy of H attached to v. Clearly, G_1 is a spanning subgraph of $G \circ H$. Thus,

$$\begin{aligned} cc(G \circ H) &\leq cc(G_2) \\ &= \sum_{v \in V(G)} cc(H_v) \\ &= \sum_{v \in V(G)} cc(H) \\ &= cc(H)|V(G)| \end{aligned}$$

Let $G_2 = \bigcup_{v_1 \in V(G)} H_{v_1}$, where H_{v_1} is a copy of H not attached to v_1 for each $v_1 \in V(G)$. Clearly, G_2 is an induced subgraph of $G \circ H$. Thus,

$$\begin{aligned} cc(G \circ H) &\geq cc(G_1) \\ &= \sum_{v_1 \in V(G)} H_{v_1} \\ &= \sum_{v_1 \in V(G)} H \\ &= cc(H)|V(G)| \end{aligned}$$

Accordingly, $cc(G \circ H) = cc(H)|V(G)|$. □

Corollary 3.4 $cc(C_n \circ K_1) = n$.

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