H-Type Group, $a$-Weyl Transform and Pseudo-Differential Operators

Mingkai Yin

School of Mathematics and Information Sciences
Guangzhou University, Guangzhou, 510006, P.R. China
Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institutes
Guangzhou University, Guangzhou, 510006, P.R. China

Jianxun He

School of Mathematics and Information Sciences
Guangzhou University, Guangzhou, 510006, P.R. China
Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institutes
Guangzhou University, Guangzhou, 510006, P.R. China

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Abstract

This paper is about the pseudo-differential operator on the H-type group, denoted by $H$. We present the trace formula for the $a$-Weyl transform. Also, we give the characterization of Hilbert-Schmidt class and trace class pseudo-differential operators on $H$.

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1 Introduction

The theory of pseudo-differential operators was introduced by Hörmander [6], Kohn-Nirenberg [9], Wong [14], and others. It can be applied to the fields of operator theory, differential equations, mathematical physics and engineering aspects. Some further development we refer readers to [2, 4]. Recently, Dasgupta and Wong [1] studied the properties of pseudo-differential operators associated with operator-valued symbols on the Heisenberg group. We are interested in pseudo-differential operators on the H-type group. The H-type group has been of substantial interest in mathematics. For example, the geometry properties, multiplier, Radon transforms and heat kernel on the H-type group were studied in [5, 7, 8, 11, 12, 16]. This paper is to investigate the theory of pseudo-differential operators on the H-type group. Our approach depends on the computation of the Weyl transform.

Let \( H \) be a H-type group. The underlying manifold is \( \mathbb{R}^{2n+m} \). For \((z, t) \in H\), \((z, t)\) can also be written as \((x, y, t)\) where \(x, y \in \mathbb{R}^n, t \in \mathbb{R}^m\). The group law is given by

\[
(z, t)(z', t') = (z + z', t + t' + \frac{1}{2} \text{Im}(zz')),
\]

where \((z, t), (z', t') \in H\), \((\text{Im}(zz'))_j = (z, U^{(j)} z')\), \(U^{(j)}\) is a \(2n \times 2n\) skew-symmetric orthogonal matrix, \(j = 1, 2, \ldots, m\). \(U^{(j)}\) satisfying \(U^{(i)}U^{(j)} + U^{(j)}U^{(i)} = 0\), \(i, j = 1, 2, \ldots, m\) with \(i \neq j\). We define the Schrödinger representation \(\pi_a\) of \(H\) by

\[
\pi_a(x, y, t) \phi(\xi) = e^{ia \cdot t + i|a|(x \cdot \xi + \frac{x \cdot y}{2})} \phi(\xi + y),
\]

where \((x, y, t) \in H\), \(a \in \mathbb{R}^{m*} = \mathbb{R}^m \setminus \{0\}\) and \(\phi \in L^2(\mathbb{R}^n)\). Thus \(\pi_a\) is the irreducible representation on \(H\). For more details we refer the reader to [10] and [16]. \(\pi_a(x, y, t)\) can also be written as \(\pi_a(x, y, t) = e^{ia \cdot t} \pi_a(x, y)\), where \(\pi_a(x, y)\) is

\[
\pi_a(x, y) \phi(\xi) = e^{i|a|(x \cdot \xi + \frac{x \cdot y}{2})} \phi(\xi + y).
\]

The Fourier transform of \(f \in L^1(\mathbb{R}^n)\) is given by

\[
\hat{f}(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-ix \cdot y} dx.
\] (1)

From [13], we know that the group Fourier transform of a function \(f \in L^1(H)\) is an operator-valued function defined by

\[
\hat{f}(a) = \int_{\mathbb{R}^{2n+m}} f(x, y, t) \pi_a(x, y, t) dx dy dt.
\]
In this section, we are going to study the trace formula for the I-transform. Similar to \([15]\), we need to define a on the H-type group. Let \(f, g \in \mathcal{S}^2\) of all Hilbert-Schmidt operators on \(L^2(\mathbb{R}^n)\). The group Fourier inversion formula is given by

\[
f(x, y, t) = \int_{\mathbb{R}^n} tr(\pi_a(x, y, t)^* \hat{f}(a)) d\mu(a),
\]

where \(\pi_a(x, y, t)^*\) is the adjoint of \(\pi_a(x, y, t)\).

\section{a-Weyl transforms}

In this section, we are going to study the trace formula for the \(a\)-Weyl transform on the H-type group. Let \(f, g \in L^2(\mathbb{R}^n)\). The \(a\)-Fourier-Wigner transform of \(f\) and \(g\) is defined by

\[
V_a(f, g)(p, q) = |a|^{n/2}(2\pi)^{-n/2} (\pi_a(p, q) f, g),
\]

where \((\cdot, \cdot)\) is the inner product in \(L^2(\mathbb{R}^n)\). Then we have

\[
V_a(f, g)(p, q) = |a|^{n/2}(2\pi)^{-n/2} (\pi_a(p, q) f, g) = |a|^{n/2}(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i|a|(p \cdot x + \frac{q \cdot x}{2})} f(x + \frac{q}{2})g(x) dx = |a|^{n/2}(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i|a|(p \cdot x)} f(x + \frac{q}{2})g(x - \frac{q}{2}) dx.
\]

We can also define the Fourier transform by

\[
(F|a|^{-1}(f))(y) = |a|^{\frac{n}{2}} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x)e^{-i|a|x\cdot y} dx,
\]

where \(a \in R^{m*}, f \in L^1(\mathbb{R}^n)\). For \(|\alpha| = 1\), the above Fourier transform is the same as \((1)\). The inverse Fourier transform is defined by

\[
(F|a|^{-1}(f))(x) = |a|^{\frac{n}{2}} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i|a|x\cdot y} f(y) dy.
\]

Now we are going to compute the Fourier transform of the \(a\)-Fourier-Wigner transform. Similar to \([15]\), we need to define \(I_{\varepsilon}\) by

\[
I_{\varepsilon}(x, \xi) = \frac{(|\alpha|^{\frac{n}{2}}e^{-\varepsilon^2|y|^2/2-|\alpha|\xi \cdot y} \int_{\mathbb{R}^n} e^{-\varepsilon^2|y|^2/2-|\alpha|\xi \cdot y} \int_{\mathbb{R}^n} e^{-\varepsilon^2|\xi|^2/2} f(y + \frac{q}{2})g(y - \frac{q}{2}) dy dq}{2\pi}.
\]

\[
= |a|^{\frac{n}{2}} (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\alpha \cdot \xi q} \int_{\mathbb{R}^n} e^{-\varepsilon^2|\xi|^2/2} f(y + \frac{q}{2})g(y - \frac{q}{2}) dy dq.
\]
As $\varepsilon$ goes to 0, we then have
\[
(F_{|a|}(V_a(f,g)))(x,\xi) = |a|^{n/2}(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-|a|\xi \cdot q} f(x + \frac{q}{2}) g(x - \frac{q}{2}) dq.
\]

Then the $a$-Wigner transform is defined by
\[
W_a(f, g)(x, \xi) = (F_{|a|}(V_a(f,g)))(x, \xi) = |a|^{n/2}(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-|a|\xi \cdot q} f(x + \frac{q}{2}) g(x - \frac{q}{2}) dq,
\]

where $f, g \in L^2(\mathbb{R}^n)$. For $|a| = 1$, the $a$-Wigner transform can also be written as
\[
W(f, g)(x, \xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi \cdot q} f(x + \frac{q}{2}) g(x - \frac{q}{2}) dq.
\]

Let $\sigma$ be a function in the Schwartz space $S(\mathbb{R}^{2n})$, we define $W_{\sigma}^a$ to be the $a$-Weyl transform associated to the symbol $\sigma$ as following
\[
\langle W_{\sigma}^a f, g \rangle = |a|^{n/2}(2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) W_a(f, g)(x, \xi) dx d\xi
\]
\[
= |a|^{n/2}(2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (F_{|a|}\sigma)(p, q) V_a(f, g)(p, q) dp dq.
\]

Thus we can also write
\[
W_{\sigma}^a = |a|^n(2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (F_{|a|}\sigma)(p, q) \pi_a(p, q) dp dq.
\]

For $|a| = 1$, the $a$-Weyl transform is given by
\[
\langle W_{\sigma} f, g \rangle = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) W(f, g)(x, \xi) dx d\xi.
\]
\[
W_{\sigma} = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{\sigma}(p, q) \pi(p, q) dp dq.
\]

We now define $D_{|a|}^1$ and $D_{|a|}^2$ by
\[
D_{|a|}^1 \sigma(x, \xi) = \sigma(|a|x, \xi)
\]
and
\[
D_{|a|}^2 \sigma(x, \xi) = \sigma(x, |a|\xi)
\]
respectively, then we have the following theorem.
**Theorem 2.1** Let $\sigma$ be a function in the Schwartz space $S(\mathbb{R}^{2n})$, then we have

\[ W_\sigma^a = W_{D_{|a|}^{-1}\sigma} = W_{D_{|a|}^{-1}\sigma}. \]

The trace formula for the a-Weyl transform is

\[
\text{tr}(W_\sigma^a) = |a|^n \text{tr}(W_\sigma) = |a|^n (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) dxd\xi.
\]

Also, we have

\[
\text{tr}(W_\sigma^a W_\tau^a) = |a|^n \text{tr}(W_\sigma W_\tau).
\]

**Proof:** By a direct computation, we have

\[
\langle W_\sigma^a f, g \rangle = |a|^{n/2} (2\pi)^{-n/2} \int_{\mathbb{R}^{2n}} \sigma(x, \xi) W_\sigma(f, g)(x, \xi) dxd\xi
\]

\[
= |a|^n (2\pi)^{-n} \int_{\mathbb{R}^{3n}} \sigma(x, \xi) (2\pi)^{-n/2} e^{-|a|\xi \cdot q} f(x + \frac{q}{2}) g(x - \frac{q}{2}) dq dxd\xi
\]

\[
= (2\pi)^{-n} \int_{\mathbb{R}^{3n}} \sigma(x, \frac{\xi}{|a|}) (2\pi)^{-n/2} e^{-\frac{q}{2} \xi \cdot q} f(x + \frac{q}{2}) g(x - \frac{q}{2}) dq dxd\xi
\]

\[
= \langle W_{D_{|a|}^{-1}\sigma} f, g \rangle.
\]

\[
W_\sigma^a f(x) = |a|^n (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (F(a) \sigma)(p, q) \pi_\sigma(p, q) f(x) dp dq
\]

\[
= |a|^{2n} (2\pi)^{-2n} \int_{\mathbb{R}^{3n}} \sigma(p, q) e^{-|a|p \cdot y} e^{-|a|q \cdot \xi} dy d\xi e^{i|\xi|(p + \frac{q}{2})} f(x + q) dp dq
\]

\[
= |a|^{2n} (2\pi)^{-2n} \int_{\mathbb{R}^{3n}} \sigma(p, q) e^{-|a|p \cdot y} e^{-\frac{q}{2} \xi \cdot q} dy d\xi e^{i\frac{q}{2}(p + \frac{q}{2})} f(x + q) \frac{dp}{|a|^n}
\]

\[
= |a|^{2n} (2\pi)^{-2n} \int_{\mathbb{R}^{3n}} \sigma(p, q) e^{-|a|p \cdot y} e^{-\frac{q}{2} \xi \cdot q} dy d\xi e^{i\frac{q}{2}(p + \frac{q}{2})} f(x + q) \frac{dp}{|a|^n}
\]

\[
= (2\pi)^{-2n} \int_{\mathbb{R}^{3n}} \sigma(p, q) e^{-|a|p \cdot y} e^{-\frac{q}{2} \xi \cdot q} dy d\xi e^{i\frac{q}{2}(p + \frac{q}{2})} f(x + q) dp dq
\]

\[
= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} D_{|a|}^{-1}\sigma(p, q) \pi(p, q) f(x) dp dq
\]

\[
= W_{D_{|a|}^{-1}\sigma} f(x).
\]

So we have $W_\sigma^a = W_{D_{|a|}^{-1}\sigma} = W_{D_{|a|}^{-1}\sigma}$. Using the trace formula in [3], we get

\[
\text{tr}(W_\sigma^a) = \text{tr}(W_{D_{|a|}^{-1}\sigma})
\]
\[
= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} D^2_{|a|-1} \sigma(x, \xi) dx d\xi
\]
\[
= |a|^n (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(x, \xi) dx d\xi
\]
\[
= |a|^n tr(W_{\sigma}).
\]

Similarly, we obtain
\[
tr(W_{\sigma}^a) = tr(W_{D^1_{|a|-1} \sigma})
\]
\[
= |a|^n tr(W_{\sigma}).
\]

We also have
\[
tr(W_{\sigma}^a W_{\tau}^a) = tr(W_{D^1_{|a|-1} \sigma} W_{D^1_{|a|-1} \tau})
\]
\[
= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} D^1_{|a|-1} \sigma(x, \xi) D^1_{|a|-1} \tau(x, \xi) dx d\xi
\]
\[
= |a|^n (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) \tau(x, \xi) dx d\xi
\]
\[
= |a|^n tr(W_{\sigma} W_{\tau}).
\]

3 Pseudo-Differential Operators

Let \(B(L^2(\mathbb{R}^n))\) denote the \(C^\ast\)-algebra of all bounded linear operators on \(L^2(\mathbb{R}^n)\). Operator-valued symbol \(\sigma\) is a mapping \(\sigma : \mathcal{H} \times \mathbb{R}^{m^*} \to B(L^2(\mathbb{R}^n))\). We define the pseudo-differential operator \(T_{\sigma} : L^2(\mathcal{H}) \to L^2(\mathcal{H})\) by
\[
T_{\sigma} f(x, y, t) = \int_{\mathbb{R}^m} tr(\pi_a(x, y, t)^* \sigma(x, y, t, a) \hat{f}(a)) d\mu(a),
\]
where \((x, y, t) \in \mathcal{H}, f\) in the Schwartz space \(S(\mathcal{H})\). Now we are going to show the boundedness of the pseudo-differential operator.

**Theorem 3.1** Let \(\sigma : \mathcal{H} \times \mathbb{R}^{m^*} \to S_2\) be an operator-valued symbol such that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \|\sigma(x, y, t, a)\|^2_{S_2} dxdydt d\mu(a) < \infty.
\]
Then the corresponding pseudo-differential operator \(T_{\sigma} : L^2(\mathcal{H}) \to L^2(\mathcal{H})\) is a bounded linear operator.

**Proof:** Let \(f \in L^2(\mathcal{H})\).
\[
\|T_{\sigma} f\|_{L^2(\mathcal{H})}
\]
\[
= \left( \int_{\mathbb{R}^{2n+m}} \left| \int_{\mathbb{R}^n} tr(\pi_a(x, y, t)^* \sigma(x, y, t, a) \hat{f}(a)) d\mu(a) \right|^2 dxdydt \right)^{\frac{1}{2}}
\]
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$\leq \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^{2n+m}} \left| tr(\pi_a(x, y, t) \sigma(x, y, t, a) \hat{f}(a)) \right|^2 dx dy dt \right)^{\frac{1}{2}} d\mu(a)$

$\leq \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^{2n+m}} \|\sigma(x, y, t, a)\|_{S_2}^2 \|\hat{f}(a)\|_{S_2}^2 dx dy dt \right)^{\frac{1}{2}} d\mu(a)$

$= \int_{\mathbb{R}^m} \|\hat{f}(a)\|_{S_2} \left( \int_{\mathbb{R}^{2n+m}} \|\sigma(x, y, t, a)\|_{S_2}^2 dx dy dt \right)^{\frac{1}{2}} d\mu(a)$

$\leq \left( \int_{\mathbb{R}^m} \|\hat{f}(a)\|_{S_2}^2 d\mu(a) \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n+m}} \|\sigma(x, y, t, a)\|_{S_2}^2 dx dy dt d\mu(a) \right)^{\frac{1}{2}}$

$= \|f\|_{L^2(H)} \left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n+m}} \|\sigma(x, y, t, a)\|_{S_2}^2 dx dy dt d\mu(a) \right)^{\frac{1}{2}}$.

Thus $T_\sigma$ is a bounded linear operator.

Now we are going to study the property about the symbol of the pseudo-differential operator.

**Theorem 3.2** Let $\sigma : H \times \mathbb{R}^{m*} \rightarrow S_2$ be an operator-valued symbol such that

$\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\sigma(x, y, t, a)\|_{S_2}^2 dx dy dt d\mu(a) < \infty,$

$\int_{\mathbb{R}^m} \|\sigma(x, y, t, a)\|_{S_2} d\mu(a) < \infty,$

$\sup_{(x, y, t, a) \in H \times \mathbb{R}^{m*}} \|\sigma(x, y, t, a)\|_{S_2}^2 < \infty,$

and the mapping

$H \times \mathbb{R}^{m*} \ni (x, y, t, a) \rightarrow \pi_a(x, y, t)^* \sigma(x, y, t, a) \in S_2$

is weakly continuous. Then $T_\sigma f = 0$ for all $f \in L^2(H)$ only if $\sigma(x, y, t, a) = 0$ for all $(x, y, t, a) \in H \times \mathbb{R}^{m*}$.

**Proof:** For $(x, y, t) \in H$, we define $f_{(x, y, t)} \in L^2(H)$ by

$f_{(x, y, t)}(a) = \sigma(x, y, t, a)^* \pi_a(x, y, t), \ a \in \mathbb{R}^{m*}$.

So we have

$(T_\sigma f_{(x, y, t)})(x', y', t')$

$= \int_{\mathbb{R}^m} tr(\pi_a(x', y', t')^* \sigma(x', y', t', a) \sigma(x, y, t, a)^* \pi_a(x, y, t)) d\mu(a),$
where \((x', y', t') \in H\). Let \((x_0, y_0, t_0) \in H\), by the weakly continuous mapping (4), we have
\[
tr(\pi_a(x', y', t')^* \sigma(x', y', t', a) \sigma(x, y, t, a)^* \pi_a(x, y, t))
\]
goesto
\[
tr(\pi_a(x_0, y_0, t_0)^* \sigma(x_0, y_0, t_0, a) \sigma(x, y, t, a)^* \pi_a(x, y, t))
\]
as \((x', y', t')\) goes to \((x_0, y_0, t_0)\). Because of (3), we can find a constant \(C\) such that for all \((x', y', t', a) \in H \times \mathbb{R}^m\),
\[
|tr(\pi_a(x', y', t')^* \sigma(x', y', t', a) \sigma(x, y, t, a)^* \pi_a(x, y, t))| \leq C \|\sigma(x, y, t, a)\|_{S_2}.
\]
By (2), we get
\[
\int_{\mathbb{R}^m} \|\sigma(x, y, t, a)\|^2_{S_2} d\mu(a) < \infty.
\]

By Lebesgue’s dominated convergence theorem, we obtain \((T_\sigma f_{(x,y,t)})(x', y', t')\) goes to \((T_\sigma f_{(x,y,t)})(x_0, y_0, t_0)\) as \((x', y', t')\) goes to \((x_0, y_0, t_0)\). So \(T_\sigma f_{(x,y,t)}\) is continuous on \(H\). Let \((x_0, y_0, t_0) = (x, y, t)\), then
\[
(T_\sigma f_{(x,y,t)})(x, y, t) = \int_{\mathbb{R}^m} tr(\pi_a(x, y, t)^* \sigma(x, y, t, a) \sigma(x, y, t, a)^* \pi_a(x, y, t)) d\mu(a)
\]
\[
= \int_{\mathbb{R}^m} tr(\sigma(x, y, t, a) \sigma(x, y, t, a)^*) d\mu(a)
\]
\[
= \int_{\mathbb{R}^m} \|\sigma(x, y, t, a)\|^2_{S_2} d\mu(a) = 0.
\]
Then \(\|\sigma(x, y, t, a)\|_{S_2} = 0\) for almost all \(a \in \mathbb{R}^{m^*}\) only if \(\sigma(x, y, t, a) = 0\) for almost all \((x, y, t, a) \in H \times \mathbb{R}^{m^*}\).

4 Hilbert-Schmidt Operators

In this section, we are going to study Hilbert-Schmidt operators and investigate the relationship between pseudo-differential operators and Hilbert-Schmidt operators.

Theorem 4.1 Let \(f \in L^1(H)\). Then for all \(a \in \mathbb{R}^{m^*}\)
\[
\hat{f}(a) = |a|^{-n} (2\pi)^{-\frac{2n+m}{2}} W^{a}_{f|a|} f^a,
\]
where \(f^a\) is defined by
\[
f^a(x, y) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^n} e^{ia \cdot t} f(x, y, t) dt.
\]
Proof: Let \( \varphi \in S(\mathbb{R}^n) \). Then we have

\[
(\hat{f}(a)\varphi)(\xi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, y, t)\pi_a(x, y, t)\varphi(\xi) dx dy dt
\]

\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, y, t)e^{ia\xi_1}\pi_a(x, y, t)\varphi(\xi) dx dy dt
\]

\[
= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f^a(x, y, t)\pi_a(x, y, t)\varphi(\xi) dx dy
\]

\[
= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mathcal{F}_{|a|}(\mathcal{F}_{|a|}^{-1}f^a))(x, y)\pi_a(x, y, t)\varphi(\xi) dx dy.
\]

So we have

\[
\hat{f}(a) = |a|^{-n}(2\pi)^{\frac{2n+m}{2}}W_{|a|}^{-1}(f^a).
\]

**Theorem 4.2** Let \( \sigma \) be a symbol that satisfies the hypotheses of Theorem 3.2. Then the corresponding pseudo-differential operator \( T_\sigma \) is a Hilbert-Schmidt operator if and only if

\[
\sigma(x, y, t, a) = |a|^{-n}\pi_a(x, y, t)W_{|a|}^{-1}(\alpha(x, y, t)^{-a}),
\]

where \( (x, y, t, a) \in H \times \mathbb{R}^{m*} \), \( \alpha : H \to L^2(H) \) is a weakly continuous mapping such that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\alpha(x, y, t)\|^2_{L^2(H)} dx dy dt < \infty,
\]

\[
\sup_{(x, y, t, a) \in H \times \mathbb{R}^{m*}} |a|^{-n/2}\|\alpha(x, y, t)^{-a}\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} < \infty
\]

and

\[
\int_{\mathbb{R}^{2n+m}} |a|^{n/2}\|\alpha(x, y, t)^{-a}\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} da < \infty.
\]

**Proof:** Firstly, we show the sufficiency. Set \( f \in S(H) \). Using the trace formula above, we have

\[
T_\sigma f(x, y, t)
\]

\[
= \int_{\mathbb{R}^m} tr(\pi_a(x, y, t)^*\sigma(x, y, t, a)\hat{f}(a))d\mu(a)
\]

\[
= (2\pi)^{-n}\int_{\mathbb{R}^m} tr(\pi_a(x, y, t)^*\sigma(x, y, t, a)\hat{f}(a))|a|^n da
\]

\[
= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^m} tr(W_{|a|}^{-n}\mathcal{F}_{|a|}(\alpha(x, y, t)^{-a})W_{|a|}^{-n}\mathcal{F}_{|a|}^{-1}(f^a))|a|^n da
\]

\[
= (2\pi)^{-\frac{2n+m}{2}} \int_{\mathbb{R}^{2n+m}} \mathcal{F}_{|a|}(\alpha(x, y, t)^{-a})(x', y'),\mathcal{F}_{|a|}^{-1}(f^a)(x', y') dx' dy' da
\]

\[
= (2\pi)^{-\frac{2n+m}{2}} \int_{\mathbb{R}^{2n+m}} \alpha(x, y, t)^{-a}(x', y')f^a(x', y') dx' dy' da
\]

\[
= (2\pi)^{-\frac{2n+m}{2}} \int_{\mathbb{R}^{2n+m}} \alpha(x, y, t)(x', y', a)f(x', y', a) dx' dy' da.
\]
The kernel of $T_\sigma$ is $k$ on $\mathbb{R}^{2n+m} \times \mathbb{R}^{2n+m}$ given by
\[
k(x, y, t, x', y', a) = (2\pi)^{-\frac{n+m}{2}} \alpha(x, y, t)(x', y', a),
\]
where $(x, y, t), (x', y', a) \in \mathbb{R}^{2n+m}$. By Fubini’s theorem and Plancherel’s theorem,
\[
\int_{\mathbb{R}^{2n+m}} \int_{\mathbb{R}^{2n+m}} |k(x, y, t, x', y', a)|^2 \, dx \, dy \, dt \, dx' \, dy' \, da
= (2\pi)^{-n-m} \int_{\mathbb{R}^{2n+m}} \|\alpha(x, y, t)(x', y', a)\|^2_{L^2(H)} \, dx \, dy \, dt < \infty.
\]
So, $T_\sigma$ is a Hilbert-Schmidt operator. Conversely, suppose that $T_\sigma$ is a Hilbert-Schmidt operator. Then there exists a function $\alpha \in L^2(\mathbb{R}^{2n+m} \times \mathbb{R}^{2n+m})$ such that for all $f \in L^2(H)$,
\[
T_\sigma f(x, y, t) = \int_{\mathbb{R}^{m+n}} \int_{\mathbb{R}^{n+m}} \alpha(x, y, t, x', y', a) f(x', y', a) \, dx' \, dy' \, d\mu(a).
\]
Let $\alpha : H \to L^2(H)$ be the mapping defined by
\[
\alpha(x, y, t)(x', y', a) = \alpha(x, y, t, x', y', a).
\]
From [15], we have
\[
\|\sigma(x, y, t, a)\|_{s_2} = (2\pi|a|)^{-n/2}\|\alpha(x, y, t)^{-a}\|_{L^2(\mathbb{R}^{2n})}.
\]
Then reversing the argument in the proof of the sufficiency and using Theorem 3.2, the converse is proved.

Now show the relationship between pseudo-differential operators on $L^2(H)$ and $a$-Weyl transforms on $L^2(\mathbb{R}^{2n+m})$. The twisting operator $T : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2m})$ is defined by
\[
(Tf)(x, y) = f(x + \frac{y}{2}, x - \frac{y}{2}), \; x, y \in \mathbb{R}^n,
\]
the inverse of $T$ is given by
\[
(T^{-1}f)(x, y) = f(\frac{x+y}{2}, x - y), \; x, y \in \mathbb{R}^n.
\]
$K_{[a]} : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n})$ is an operator defined by
\[
(K_{[a]}f)(x, y) = (T^{-1}F^2_{[a]}f)(y, x), \; x, y \in \mathbb{R}^n, \; a \in \mathbb{R}^m,
\]
where $F^2_{[a]}$ is the Fourier transform with respect to the second variables. From [15], we have the following theorems.
Theorem 4.3 Let $\sigma \in L^2(\mathbb{R}^{2n})$. Then $W_\sigma^a$ is a Hilbert-Schmidt operator with kernel $|a|^\frac{n}{2}(2\pi)^{-\frac{n}{2}} K_{|a|}\sigma$. More precisely,

\[
(W_\sigma^a f)(x) = |a|^\frac{n}{2}(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_{|a|}\sigma(x, y)f(y)dy.
\]

Theorem 4.4 Let $\tau \in L^2(\mathbb{R}^{2n+m} \times \mathbb{R}^{2n+m})$. Then

\[
W_\tau^a = T_\sigma,
\]

where $\sigma : H \times \mathbb{R}^{m*} \to S_2$ is a symbol such that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\sigma(x, y, t, a)\|_{S_2}^2 dx dy dt d\mu(a) < \infty,
\]

\[
\sigma(x, y, t, a) = |a|^{-n} \pi_a(x, y, t) W_{\tau_{|a|}(\alpha(x, y, t)^{-a})}^a,
\]

\[
\alpha(x, y, t)(x', y', t') = |a|^{\frac{2n+m}{2}} K_{|a|}\tau(x, y, t, x', y', t').
\]

Conversely, let $\sigma : H \times \mathbb{R}^{m*} \to S_2$ be a symbol such that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\sigma(x, y, t, a)\|_{S_2}^2 dx dy dt d\mu(a) < \infty,
\]

\[
\sigma(x, y, t, a) = |a|^{-n} \pi_a(x, y, t) W_{\tau_{|a|}(\alpha(x, y, t)^{-a})}^a,
\]

where $\alpha : H \to L^2(H)$ satisfies the following condition

\[
\int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \|\alpha(x, y, t)\|_{S_2}^2 dx dy dt < \infty.
\]

Then $T_\sigma = W_\tau^a$, where

\[
\tau = |a|^{-\frac{2n+m}{2}} K_{|a|}^{-1} \tilde{\alpha},
\]

$\tilde{\alpha}$ is a function on $\mathbb{R}^{2n+m} \times \mathbb{R}^{2n+m}$ given by

\[
\tilde{\alpha}(x, y, t, x', y', t') = \alpha(x, y, t)(x', y', t').
\]
5 Trace class Operators

Now we are going to study trace class pseudo-differential operators. From Theorem 4.2, we can get the expression of the kernel of pseudo-differential operators. Since a trace class operator is a product of two Hilbert-Schmidt operators, we have the following theorem.

**Theorem 5.1** Let $\sigma : H \times \mathbb{R}^{m^*} \rightarrow S_2$ be a symbol satisfying the conditions of Theorem 3.2. Then $T_{\sigma}$ is a trace class operator if and only if

$$\sigma(x, y, t, a) = |a|^{-n} \pi \alpha_1(x, y, t) W^a \mathcal{F}_{|a|}(\alpha(x, y, t) - a),$$

where $\alpha : H \rightarrow L^2(H)$ is a mapping such that the conditions of Theorem 4.2 are satisfied and

$$\alpha(x, y, t)(x', y', t') = \int_{\mathbb{R}^{2n+m}} \alpha_1(x, y, t)(x'', y'', t'') \alpha_2(x'', y'', t'')(x', y', t') dx'' dy'' dt'',$$

$\alpha_1 : H \rightarrow L^2(H)$ satisfies

$$\int_{\mathbb{R}^{2n+m}} \|\alpha_1(x, y, t)\|^2_{L^2(H)} dx dy dt < \infty,$$

$\alpha_2 : H \rightarrow L^2(H)$ satisfies

$$\int_{\mathbb{R}^{2n+m}} \|\alpha_2(x, y, t)\|^2_{L^2(H)} dx dy dt < \infty.$$

If $T_{\sigma} : L^2(H) \rightarrow L^2(H)$ is a trace class operator, then we have the trace formula

$$\text{tr}(T_{\sigma}) = \int_{\mathbb{R}^{2n+m}} \alpha(x, y, t)(x, y, t) dx dy dt$$

$$= \int_{\mathbb{R}^{4n+2m}} \alpha_1(x, y, t)(x'', y'', t'') \alpha_2(x'', y'', t'')(x, y, t) dx'' dy'' dt'' dx dy dt.$$

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**References**


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