Results On Almost $\gamma$-Continuity

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Abstract

In this paper, we introduce the concept of almost $\gamma$-continuity. And we study characterizations of such functions and relationships between almost $\gamma$-continuity and weakly $\gamma$-continuity.

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1. INTRODUCTION

Let $X$ and $Y$ be topological spaces on which no separation axioms are assumed unless explicit stated. Let $S$ be a subset of $X$. The closure (resp. interior) of $S$ will be denoted by $cl(S)$ (resp. $int(S)$). A subset $S$ of $X$ is called semi-open set [4] (resp. $\alpha$-set, $\beta$-open set [8], preopen set [5]) if $S \subseteq cl(int(S))$
(resp. $S \subseteq \text{int}(\text{cl}(\text{int}(S)))$, $S \subseteq \text{cl}(\text{int}(\text{cl}(S)))$, $S \subseteq \text{int}(\text{cl}(S)))$. The complement of a semi-open set (resp. $\alpha$-set, $\beta$-open set, preopen set) is called semi-closed set (resp. $\alpha$-closed set, $\beta$-closed set, preclosed set).

A subset $M(x)$ of a space $X$ is called a semi-neighborhood of a point $x \in X$ if there exists a semi-open set $S$ such that $x \in S \subseteq M(x)$. In [1], Latif introduced the notion of semi-convergence of filters. And he investigated some characterizations related to semi-open continuous functions. Now we recall the concept of semi-convergence of filters. Let $S(x) = \{A \in SO(X) : x \in A\}$ and let $S_x = \{A \subseteq X : \text{there exists } \mu \subseteq S(x) \text{ such that } \mu \text{ is finite and } \cap \mu \subseteq A\}$. Then $S_x$ is called the semi-neighborhood filter at $x$. For any filter $F$ on $X$, we say that $F$ semi-converges to $x$ if and only if $F$ is finer than the semi-neighborhood filter at $x$. A subset $U$ of $X$ is called a $\gamma$-set [6] in $X$ if whenever a filter $F$ semi-converges to $x$ and $x \in U$, then $U \in F$. The class of all $\gamma$-sets in $X$ will be denoted by $\gamma(X)$.

The $\gamma$-interior [6] of a set $A$ in $X$, denoted by $I_\gamma(A)$, is the union of all $\gamma$-sets contained in $A$.

The $\gamma$-closure [6] of a set $A$ in $X$, denoted by $\text{Cl}_\gamma(A)$, $\text{Cl}_\gamma(A) = \{x \in X : A \cap U \neq \emptyset \text{ for all } U \in S_x\}$.

**Theorem 1.1** ([6]). Let $(X, \tau)$ be a topological space and $A \subseteq X$.
(a) $I_\gamma(A) = \{x \in A : A \in S_x\}$.
(b) $A$ is $\gamma$-set if and only if $A = I_\gamma(A)$.
(c) A set $A$ is $\gamma$-closed if and only if whenever $F$ semi-converges to $x$ and $A \in F$, then $x \in A$.

**Theorem 1.2** ([6]). Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$.
(1) $A \subseteq \text{Cl}_\gamma(A)$.
(2) $A$ is $\gamma$-closed if and only if $A = \text{Cl}_\gamma A$.
(3) $I_\gamma(A) = X - \text{Cl}_\gamma(X - A)$.
(4) $\text{Cl}_\gamma(A) = X - I_\gamma(X - A)$.

**Definition 1.3.** Let $f : (X, \tau) \to (Y, \mu)$ be a function on two topological spaces. Then
(1) $f$ is said to be weakly $\gamma$-continuous [7] if for $x \in X$ and each open subset $V$ containing $f(x)$, there is a $\gamma$-set $U$ containing $x$ such that $f(U) \subseteq \text{cl}(V)$.
(2) $f$ is said to be $\gamma$-continuous [6] if the inverse image of each open set of $Y$ is a $\gamma$-set in $X$.

In [2], Latif showed that $f$ is $\gamma$-continuous if and only if for $x \in X$ and each open subset $V$ containing $f(x)$, there is a $\gamma$-set $U$ containing $x$ such that $f(U) \subseteq V$. 

2. Almost $\gamma$-Continuous Functions

**Definition 2.1.** Let $(X, \tau)$ and $(Y, \mu)$ be two topological spaces. Then $f : X \to Y$ is said to be almost $\gamma$-continuous at $x \in X$ if for each open subset $V$ containing $f(x)$, there is a $\gamma$-set $U$ containing $x$ such that $f(U) \subseteq \text{int}(\text{cl}(V))$. A function $f : (X, \tau) \to (Y, \mu)$ is said to be almost $\gamma$-continuous if it has the property at each point of $X$.

We get the following implications but the converses are not true:

$$\text{continuous} \Rightarrow \gamma\text{-continuous} \Rightarrow \text{almost } \gamma\text{-continuous} \Rightarrow \text{weakly } \gamma\text{-continuous}$$

**Example 2.2.** Let $X = \{a, b, c, d\}$.

(1) Consider a topology $\tau = \{\emptyset, \{a, b\}, \{a, b, d\}, X\}$ on $X$ and a function $f : (X, \tau) \to (X, \tau)$ defined as follows: $f(a) = f(c) = f(d) = d$ and $f(b) = b$. Then $f$ is almost $\gamma$-continuous but not $\gamma$-continuous.

(2) Consider a topology $\tau = \{\emptyset, \{d\}, \{a, b\}, \{a, b, d\}, X\}$ on $X$ and a function $f : (X, \tau) \to (X, \tau)$ defined as follows: $f(a) = f(c) = c$ and $f(b) = f(d) = d$. Then $f$ is weakly $\gamma$-continuous. But $f$ is not almost $\gamma$-continuous at $b$ because of $\text{int}(\text{cl}(\{d\})) = \{d\}$. Hence $f$ is not almost $\gamma$-continuous.

**Theorem 2.3.** Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces $(X, \tau)$ and $(Y, \mu)$. Then the following statements are equivalent:

(1) $f$ is almost $\gamma$-continuous at $x \in X$.

(2) $x \in I_\gamma(f^{-1}(\text{int}(\text{cl}(V))))$ for every open set $V$ containing $f(x)$.

(3) $x \in I_\gamma(f^{-1}(\text{sCl}(V)))$ for every open set $V$ containing $f(x)$.

(4) $x \in I_\gamma(f^{-1}(\text{cl}(V)))$ for every regular open set $V$ containing $f(x)$.

(5) For every regular open set $V$ containing $f(x)$, there exists a $\gamma$-set $U$ containing $x$ such that $f(U) \subseteq V$.

**Proof.** (1) $\Rightarrow$ (2) Let $V$ be an open set of $Y$ containing $f(x)$. There exists a $\gamma$-set $U$ of $X$ containing $x$ such that $f(U) \subseteq \text{int}(\text{cl}(V))$. Since $x \in U \subseteq f^{-1}(\text{int}(\text{cl}(V)))$, by definition of $\gamma$-interior, it is $x \in I_\gamma(f^{-1}(\text{int}(\text{cl}(V))))$. (2) $\Rightarrow$ (3) From $\text{int}(\text{cl}(V)) \subseteq \text{sCl}(V)$ and (2), it follows $x \in I_\gamma(f^{-1}(\text{sCl}(V)))$.

(3) $\Rightarrow$ (4) Let $V$ be any regular open set of $Y$ containing $f(x)$. Since $V = \text{int}(\text{cl}(V)) = \text{sCl}(V)$, by (3), we have $x \in I_\gamma(f^{-1}(V))$.

(4) $\Rightarrow$ (5) it is obvious.

(5) $\Rightarrow$ (1) Let $V$ be an open set of $Y$ containing $f(x)$. Then $f(x) \in V \subseteq \text{int}(\text{cl}(V))$. Since $\text{int}(\text{cl}(V))$ is regular open, there exists a $\gamma$-set $U$ containing $x$ such that $f(x) \in f(U) \subseteq \text{int}(\text{cl}(V))$. Hence $f$ is almost $\gamma$-continuous at $x$. $\square$

**Theorem 2.4.** Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces $(X, \tau)$ and $(Y, \mu)$. Then the following statements are equivalent:

(1) $f$ is almost $\gamma$-continuous.

(2) $f^{-1}(V) \subseteq I_\gamma(f^{-1}(\text{int}(\text{cl}(V))))$ for every open subset $V$ of $Y$.

(3) $\text{Cl}_\gamma(f^{-1}(\text{cl}(\text{int}(F)))) \subseteq f^{-1}(F)$ for every closed set $F$ of $Y$. 
(4) $\text{Cl}_\gamma(f^{-1}(\text{cl}(\text{int}(\text{cl}(B))))) \subseteq f^{-1}(\text{cl}(B))$ for every set $B$ of $Y$.
(5) $f^{-1}(\text{int}(B)) \subseteq I_\gamma(f^{-1}(\text{int}(\text{cl}(B))))$ for every set $B$ of $Y$.
(6) $f^{-1}(V) = I_\gamma(f^{-1}(V))$ for every regular open subset $V$ of $Y$.

Proof. (1) $\Rightarrow$ (2) Let $V$ be an open set in $Y$ and $x \in f^{-1}(V)$. There exists a $\gamma$-set $U$ of $X$ containing $x$ such that $f(U) \subseteq \text{int}(\text{cl}(V))$. Since $x \in U \subseteq f^{-1}(\text{int}(\text{cl}(V)))$, by definition of $\gamma$-interior, $x \in I_\gamma(f^{-1}(\text{int}(\text{cl}(V))))$. Hence $f^{-1}(V) \subseteq I_\gamma(f^{-1}(\text{int}(\text{cl}(V))))$.

(2) $\Rightarrow$ (3) Let $F$ be a closed subset in $Y$. Then $Y - F$ in open in $Y$. By (2),
$$
\begin{align*}
  f^{-1}(Y - F) & \subseteq I_\gamma(f^{-1}(\text{int}(Y - F))) \\
  & = I_\gamma(f^{-1}(Y - \text{cl}(F))) \\
  & \subseteq X - \text{Cl}_\gamma(f^{-1}(\text{int}(\text{cl}(F)))).
\end{align*}
$$
Thus $\text{Cl}_\gamma(f^{-1}(\text{int}(F)))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) Let $B$ be a subset of $Y$. Since $\text{cl}(B)$ is closed in $Y$, from (3), it follows $\text{Cl}_\gamma(f^{-1}(\text{int}(\text{cl}(B)))) \subseteq f^{-1}(\text{cl}(B))$.

(4) $\Rightarrow$ (5) Let $B$ be a subset of $Y$. Then from (4), it follows
$$
\begin{align*}
  f^{-1}(\text{int}(B)) & = X - f^{-1}(\text{cl}(Y - B)) \\
  & \subseteq X - \text{Cl}_\gamma(f^{-1}(\text{int}(\text{cl}(Y - B)))) \\
  & = I_\gamma(f^{-1}(\text{int}(\text{cl}(B))))).
\end{align*}
$$
Thus we get the result.

(5) $\Rightarrow$ (6) Let $V$ be any regular open subset of $Y$. By (5), $f^{-1}(V) \subseteq I_\gamma(f^{-1}(V))$. Hence $f^{-1}(V) = I_\gamma(f^{-1}(V))$.

(6) $\Rightarrow$ (1) Let $x \in X$ and $V$ any regular open set in $Y$ containing $f(x)$. By (6), it is $x \in f^{-1}(V) = I_\gamma(f^{-1}(V))$. So there exists a $\gamma$-set $U$ containing $x$ such that $U \subseteq f^{-1}(V)$. Hence from Theorem 2.3 (5), $f$ is almost $\gamma$-continuous.

\[ \square \]

Theorem 2.5. Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces $(X, \tau)$ and $(Y, \mu)$. Then the following statements are equivalent:

(1) $f$ is almost $\gamma$-continuous.
(2) $f^{-1}(K) = \text{Cl}_\gamma(f^{-1}(K))$ for every regular closed set $K$ of $Y$.
(3) $\text{Cl}_\gamma(f^{-1}(G)) \subseteq f^{-1}(\text{cl}(G))$ for every $\beta$-open set $G$ of $Y$.
(4) $\text{Cl}_\gamma(f^{-1}(G)) \subseteq f^{-1}(\text{cl}(G))$ for every semiopen set $G$ of $Y$.

Proof. (1) $\Leftrightarrow$ (2) By Theorem 2.4 (6), it is obvious.

(2) $\Rightarrow$ (3) Let $G$ be any $\beta$-open set. From $\text{cl}(G) \subseteq \text{cl}(\text{int}(\text{cl}(G))) \subseteq \text{cl}(G)$, it follows $\text{cl}(G)$ is regular closed. From (2), it follows
$$
\text{Cl}_\gamma(f^{-1}(G)) \subseteq \text{Cl}_\gamma(f^{-1}(\text{cl}(G))) = f^{-1}(\text{cl}(G)).
$$
Hence $\text{Cl}_\gamma(f^{-1}(G)) \subseteq f^{-1}(\text{cl}(G))$.

(3) $\Rightarrow$ (4) It is obvious since every semiopen set is $\beta$-open.
Theorem 2.5, it is obvious.

Thus $Cl_f(f^{-1}(V)) = f^{-1}(V)$. □

**Theorem 2.6.** Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces $(X, \tau)$ and $(Y, \mu)$. Then the following statements are equivalent:

1. $f$ is almost $\gamma$-continuous.
2. $Cl_f(f^{-1}(G)) \subseteq f^{-1}(cl(G))$ for every preopen set $G$ of $Y$.
3. $f^{-1}(G) \subseteq I_f(f^{-1}(int(cl(G))))$ for every preopen set $G$ of $Y$.

**Proof.** (1) $\Leftrightarrow$ (2) Let $G$ be any preopen set in $Y$. Then since $G$ is $\beta$-open, from Theorem 2.5, it is obvious.

(1) $\Rightarrow$ (3) Let $G$ be any preopen set of $Y$. Then since $int(cl(G))$ is regular open in $Y$, from Theorem 2.4 (6), it follows $f^{-1}(G) \subseteq f^{-1}(int(cl(G))) = I_f(f^{-1}(int(cl(G))))$. Hence $f^{-1}(G) \subseteq I_f(f^{-1}(int(cl(G))))$.

(3) $\Rightarrow$ (1) Let $G$ be any regular open set in $Y$. Then since $G$ is preopen and (3), it follows that $f^{-1}(G) \subseteq I_f(f^{-1}(int(cl(G)))) = I_f(f^{-1}(G))$. Thus $f^{-1}(G) = I_f(f^{-1}(G))$. □

We recall that a subset $A$ in a topological space $X$ is said to be $\delta$-open [12] if for each $x \in A$ there exists a regular open set $G$ such that $x \in G \subseteq A$. A point $x \in X$ is called a $\delta$-cluster point of $A$ if $A \cap int(cl(V)) \neq \emptyset$ for every open set $V$ containing $x$. The set of all $\delta$-cluster points of $A$ is called $\delta$-closure of $A$ [12] and is denoted by $Cl_\delta(A)$. If $A = Cl_\delta(A)$, then $A$ is called $\delta$-closed. The complement of a $\delta$-closed set is said to be $\delta$-open. It is shown in [12] that $cl(A) = Cl_\delta(A)$ for every open set $A$ and $Cl_\delta(B)$ is closed for every subset $B$ of $X$. The set $\{x \in X : x \in U \subseteq A$ for some regular open set $U$ of $X\}$ is called the $\delta$-interior of $A$ and is denoted by $I_\delta(A)$.

**Theorem 2.7.** Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces $(X, \tau)$ and $(Y, \mu)$. Then the following statements are equivalent:

1. $f$ is almost $\gamma$-continuous.
2. $Cl_f(f^{-1}(cl(int(Cl_\delta(B)))))) \subseteq f^{-1}(Cl_\delta(B))$ for every set $B$ of $Y$.
3. $Cl_f(f^{-1}(cl(int(Cl_\delta(B)))))) \subseteq f^{-1}(Cl_\delta(B))$ for every set $B$ of $Y$.
4. $Cl_f(f^{-1}(cl(int(Cl_\delta(G)))))) \subseteq f^{-1}(cl(G))$ for every open set $G$ of $Y$.
5. $Cl_f(f^{-1}(cl(int(Cl_\delta(G)))))) \subseteq f^{-1}(cl(G))$ for every preopen set $G$ of $Y$.

**Proof.** (1) $\Rightarrow$ (2) Let $B$ be any subset in $Y$; then $Cl_\delta(B)$ is closed, by Theorem 2.4 (3), we get the result.

(2) $\Rightarrow$ (3) It is obvious since $cl(B) \subseteq Cl_\delta(B)$ for every subset $B$ of $Y$.

(3) $\Rightarrow$ (4) It is obvious since $cl(G) = Cl_\delta(G)$ for every open subset $G$ of $Y$. 

(4) $\Rightarrow$ (2) Let $V$ be any regular closed set of $Y$. Since $V$ is semiopen, by (4), $Cl_f(f^{-1}(V)) \subseteq f^{-1}(cl(V)) = f^{-1}(V)$. Thus $Cl_f(f^{-1}(V)) = f^{-1}(V)$. □
(4) $\Rightarrow$ (5) Let $G$ be a preopen subset of $Y$. Then $cl(G) = cl(int(cl(G)))$.\linebreak Set $A = int(cl(G))$ then by (4), $Cl_\gamma(f^{-1}(cl(int(cl(A)))) \subseteq f^{-1}(cl(A))$. Since $cl(A) = cl(G)$, we have $Cl_\gamma(f^{-1}(cl(int(cl(G)))) \subseteq f^{-1}(cl(G))$.

(5) $\Rightarrow$ (1) Let $A$ be a regular closed subset of $Y$. Then $int(A)$ is preopen and from (5), it follows
\[
Cl_\gamma(f^{-1}(A)) = Cl_\gamma(f^{-1}(cl(int(A))))
\]
\[
= Cl_\gamma(f^{-1}(cl(int(cl(A))))))
\]
\[
\subseteq f^{-1}(cl(int(A)))
\]
\[
= f^{-1}(A).
\]

Hence $f$ is almost $\gamma$-continuous by Theorem 2.5 (2). \hfill $\square$

**Theorem 2.8.** Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces $(X, \tau)$ and $(Y, \mu)$. Then the following statements are equivalent:

1. $f$ is almost $\gamma$-continuous.
2. $f(Cl_\gamma(B)) \subseteq Cl_\delta(f(B))$ for every set $B$ of $X$.
3. $f^{-1}(F) = Cl_\gamma(f^{-1}(F))$ for every $\delta$-closed set $F$ of $Y$.
4. $f^{-1}(G) = I_\gamma(f^{-1}(G))$ for every $\delta$-open set $G$ of $Y$.
5. $f^{-1}(I_\delta(B)) \subseteq I_\gamma(f^{-1}(B))$ for every set $B$ of $Y$.
6. $Cl_\gamma(f^{-1}(B)) \subseteq f^{-1}(Cl_\delta(B))$ for every set $B$ of $Y$.

**Proof.** (1) $\Rightarrow$ (2) Let $B$ be any subset in $Y$. Let $x \in Cl_\gamma(B)$ and $V$ any open set of $Y$ containing $f(x)$. By almost $\gamma$-continuity, there exists a $\gamma$-set $U$ containing $x$ such that $f(U) \subseteq int(cl(V))$. Since $x \in Cl_\gamma(B)$, $B \cap U \neq \emptyset$ and so $\emptyset \neq f(U) \cap f(B) \subseteq int(cl(V)) \cap f(B)$. Hence $f(x) \in Cl_\delta(f(B))$.

(2) $\Rightarrow$ (3) Let $F$ be any $Cl_\delta$-closed set of $Y$. Then from (2), it follows $f(Cl_\gamma(f^{-1}(F))) \subseteq Cl_\delta(f^{-1}(F))) \subseteq Cl_\delta(F) = F$. Hence $f^{-1}(F) = Cl_\gamma(f^{-1}(F))$.

(3) $\Rightarrow$ (4) It is obvious.

(4) $\Rightarrow$ (5) Let $B$ be any subset of $Y$. Then $I_\delta(B)$ is a $\delta$-open set of $Y$. From (4), it follows $f^{-1}(I_\delta(B)) = I_\gamma(f^{-1}(I_\delta(B))) \subseteq I_\gamma(f^{-1}(B))$. Hence we have $f^{-1}(I_\delta(B)) \subseteq I_\gamma(f^{-1}(B))$.

(5) $\Rightarrow$ (6) Let $B$ be any subset of $Y$. From (5), it follows $f^{-1}(Cl_\delta(B)) = X - f^{-1}(I_\delta(Y - B)) \supseteq X - I_\gamma(f^{-1}(Y - B))) = Cl_\gamma(f^{-1}(B))$. Hence we have $Cl_\gamma(f^{-1}(B)) \subseteq f^{-1}(Cl_\delta(B))$.

(6) $\Rightarrow$ (1) Let $B$ be any subset of $Y$. Since $Cl_\delta(B)$ is closed in $Y$, by (6), we have
\[
Cl_\gamma(f^{-1}(cl(int(Cl_\delta(B)))))) \subseteq f^{-1}(Cl_\delta(cl(int(Cl_\delta(B))))))
\]
\[
= f^{-1}(cl(int(Cl_\delta(B))))
\]
\[
\subseteq f^{-1}(Cl_\delta(B)).
\]

Hence by Theorem 2.7 (2), $f$ is almost $\gamma$-continuous. \hfill $\square$
Definition 2.9 ([7]). Let \( X \) be a topological space. Then \( X \) is said to be \( \gamma-T_2 \) if for every two distinct points \( x \) and \( y \) in \( X \), there exist two disjoint \( \gamma \)-sets \( U \) and \( V \) such that \( \text{cl}(U) \cap \text{cl}(V) = \emptyset \).

Let \( X \) be a topological space. Then \( X \) is said to be Urysohn if for every two distinct points \( x \) and \( y \) in \( X \), there exist two open sets \( U \) and \( V \) such that \( \text{cl}(U) \cap \text{cl}(V) = \emptyset \).

Theorem 2.10. Let \( f : (X, \tau) \to (Y, \mu) \) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). If \( f \) is an almost \( \gamma \)-continuous injection and \( Y \) is Urysohn, then \( X \) is \( \gamma-T_2 \).

Proof. Let \( x_1 \) and \( x_2 \) be two distinct elements in \( X \), then \( f(x_1) \neq f(x_2) \). There exist two open sets \( U \) and \( V \) in \( Y \) containing \( f(x_1) \), \( f(x_2) \), respectively, such that \( \text{cl}(U) \cap \text{cl}(V) = \emptyset \). Since \( f \) is almost \( \gamma \)-continuous, there exist \( \gamma \)-sets \( U_1, V_2 \) containing \( x_1, x_2 \), respectively, such that \( f(U_1) \subseteq \text{int}(\text{cl}(U)), f(V_2) \subseteq \text{int}(\text{cl}(V)) \). From injectivity, it follows \( U_1 \cap V_2 = \emptyset \). Hence \( X \) is \( \gamma-T_2 \).

Definition 2.11 ([7]). Let \( f : (X, \tau) \to (Y, \mu) \) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). We call \( f \) has a strongly \( \gamma \)-closed graph if for each \((x, y) \notin G(f)\), there exist a \( \gamma \)-set \( U \) and an open set \( V \) containing \( x \) and \( y \), respectively, such that \((U \times \text{cl}(V)) \cap G(f) = \emptyset\).

Lemma 2.12. Let \( f : (X, \tau) \to (Y, \mu) \) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). Then \( f \) has a strongly \( \gamma \)-closed graph if for each \((x, y) \notin G(f)\), there exist a \( \gamma \)-set \( U \) containing and an open set \( V \) containing \( x \) and \( y \), respectively, such that \( f(U) \cap \text{cl}(V) = \emptyset \).

Theorem 2.13. Let \( f : (X, \tau) \to (Y, \mu) \) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). If \( f \) is almost \( \gamma \)-continuous and \( Y \) is \( T_2 \), then \( f \) has a strongly \( \gamma \)-closed graph.

Proof. Let \((x, z) \notin G(f)\). Then \( z \neq f(x) \) and since \( Y \) is \( T_2 \), there exist two open sets \( U \) and \( V \) containing \( z \) and \( f(x) \), respectively, such that \( U \cap V = \emptyset \). Since \( f \) is almost \( \gamma \)-continuous, there exists a \( \gamma \)-set \( H \) containing \( x \) such that \( f(H) \subseteq \text{int}(\text{cl}(V)) \). It implies \( f(H) \cap \text{cl}(U) = \emptyset \). Hence by Lemma 2.12, \( f \) has a strongly \( \gamma \)-closed graph.

Theorem 2.14. Let \( f : (X, \tau) \to (Y, \mu) \) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). If \( f \) is an almost \( \gamma \)-continuous injection with a strongly \( \gamma \)-closed graph, then \( X \) is \( \gamma-T_2 \).

Proof. Let \( x_1 \) and \( x_2 \) be two distinct elements in \( X \), then \( f(x_1) \neq f(x_2) \). This implies that \((x_1, f(x_2)) \in (X \times Y) - G(f)\). Since \( f \) has a strongly \( \gamma \)-closed graph, by Lemma 2.12, there exist a \( \gamma \)-set \( U \) and an open set \( V \) containing \( x_1 \) and \( f(x_2) \), respectively, such that \( f(U) \cap \text{cl}(V) = \emptyset \). Since \( f \) is almost \( \gamma \)-continuous, there exists a \( \gamma \)-set \( W \) containing \( x_2 \) such that \( f(W) \subseteq \text{int}(\text{cl}(V)) \). It implies \( f(W) \cap f(U) = \emptyset \). Therefore \( W \cap U = \emptyset \) and so \( X \) is a \( \gamma-T_2 \) space.
A topological space \((X, \tau)\) is said to be nearly compact [11] if every collection \(\{U_i : i \in J\}\) of open subsets of \(X\) such that \(A \subset \bigcup \{U_i : i \in J\}\), there exists a finite subset \(J_0\) of \(J\) such that \(X = \bigcup \{\text{int}(\text{cl}(U_i)) : i \in J_0\}\).

**Theorem 2.15.** Let \(f : (X, \tau) \to (Y, \mu)\) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). If \(f\) is a almost \(\gamma\)-continuous surjection and \(X\) is \(\gamma\)-compact, then \(Y\) is nearly compact.

**Proof.** Let \(\{V_i : i \in J\}\) be an open cover of \(Y\). For each \(x \in X\), there exists \(i(x) \in J\) such that \(f(x) = y \in V_{i(x)}\). Since \(f\) is almost \(\gamma\)-continuous, there exists a \(\gamma\)-set \(U(x)\) containing \(x\) such that \(f(U(x)) \subseteq \text{int}(\text{cl}(V_{i(x)}))\). The family \(\{U(x) : x \in A\}\) is a cover of \(X\) by \(\gamma\)-sets in \(X\). Since \(X\) is \(\gamma\)-compact, there is a finite subcover \(\{U(x_1), U(x_2), \ldots, U(x_n) : x_j \in X, j = 1, 2, \ldots, n\}\) such that \(X \subseteq \bigcup U(x_j)\). Then

\[
Y \subseteq f(\bigcup U(x_j)) = \bigcup f(U(x_j)) \subseteq \bigcup \text{int}(\text{cl}(V_{i(x_j)})�,
\]

\[1 \leq j \leq n.\]

Thus \(Y\) is nearly compact. \(\square\)

**Definition 2.16 ([9]).** A topological space \((X, \tau)\) is said to be semi-regular if for each open set \(U\) of \(X\) and each point \(x \in U\) there exists a regular open set \(V\) of \(X\) such that \(x \in V \subseteq U\).

**Theorem 2.17.** Let \(f : (X, \tau) \to (Y, \mu)\) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). If \(f\) is almost \(\gamma\)-continuous and \(Y\) is semi-regular, then \(f\) is \(\gamma\)-continuous.

**Proof.** Let \(x \in X\) and \(U\) be an open set in \(Y\) containing \(f(x)\). By the semi-regularity of \(Y\), there exists a regular open \(V\) of \(Y\) such that \(f(x) \in V \subseteq U\). By the almost \(\gamma\)-continuity, there exists a \(\gamma\)-set \(G\) containing \(x\) such that \(f(G) \subseteq \text{int}(\text{cl}(V)) = V \subseteq U\). Hence \(f\) is \(\gamma\)-continuous. \(\square\)

**Definition 2.18.** Let \(f : (X, \tau) \to (Y, \mu)\) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). Then \(f\) is said to be almost \(\gamma\)-open if \(f(U) \subseteq \text{int}(\text{cl}(f(U)))\), for each \(\gamma\)-set \(U\) of \(X\).

**Theorem 2.19.** Let \(f : (X, \tau) \to (Y, \mu)\) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). If \(f\) is weakly \(\gamma\)-continuous and almost \(\gamma\)-open, then \(f\) is almost \(\gamma\)-continuous.

**Proof.** Let \(x \in X\) and \(U\) an open set in \(Y\) containing \(f(x)\). By the weakly \(\gamma\)-continuity, there exists a \(\gamma\)-set \(V\) of \(Y\) such that \(f(V) \subseteq \text{cl}(U)\). Since \(f\) is almost \(\gamma\)-open, \(f(V) \subseteq \text{int}(\text{cl}(f(V))) \subseteq \text{int}(\text{cl}(U))\). Thus \(f(V) \subseteq \text{int}(\text{cl}(U))\). \(\square\)

**Definition 2.20 ([10]).** A topological space \((X, \tau)\) is said to be almost-regular if for each regular closed set \(F\) of \(X\) and each point \(x \in X - F\), there exist disjoint open sets \(U\) and \(V\) of \(X\) such that \(x \in U\) and \(F \subseteq V\).
Theorem 2.21. Let \( f : (X, \tau) \to (Y, \mu) \) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). If \( f \) is weakly \( \gamma \)-continuous and \( Y \) is almost-regular, then \( f \) is almost \( \gamma \)-continuous.

Proof. Let \( x \in X \) and \( U \) be open set in \( Y \) containing \( f(x) \). By the almost-regularity of \( Y \), there exists a regular open \( G \) of \( Y \) such that \( f(x) \in G \subseteq \text{cl}(G) \subseteq \text{int}(\text{cl}(U)) \). And since \( G \) is an open set containing \( f(x) \), by the weakly \( \gamma \)-continuity, there exists a \( \gamma \)-set \( V \) containing \( x \) such that \( f(V) \subseteq \text{cl}(G) \). This implies \( f(V) \subseteq \text{int}(\text{cl}(U)) \). Hence \( f \) is almost \( \gamma \)-continuous.

\[ \square \]

References


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