The Evaluation of Integrals Involving Sine and Cosine on a Simple Closed Path

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Abstract

Residues are used in an neat and practical integration method. In this article, we have checked the evaluation of integrals involving sine and cosine on a simple closed path, and the proposed idea results from residue integration in complex variables.

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1 Introduction

Complex integration has been advanced by a reason of being evaluated by the methods of it in many complicated real and complex integrals in applications. This complex integration makes us evaluate certain real integrals appearing in applications that are not accessible by real integral calculus[3], and also plays a role in connection with gamma function and the error one. The main methods of complex integration are Cauchy’s integral formula and integration by residue. The coefficient of the power $1/(z - z_0)$ of Laurent series is called
the residue of $f(z)$ at $z_0$, and residues are used in a nice integration method called residue integration.


In this article, we would like to apply the concept of residue integration to real integral involving sine and cosine. In [1], we have probed the validity checking on the exchange of integral and limit in the solving process of PDEs, and here, the proposed theorem shows that an integral of differentiable function on a simple close path can be expressed by the limit of singular points.

2 The evaluation of integrals involving sine and cosine

In this section, we would like to deal with integrals involving sine and cosine on a simple closed path.

**Lemma 2.1 (Cauchy’s integral formula)** Let $f(z)$ be analytic in a simply connected domain $D$. Then for any point $z_0$ in $D$ and any simply closed path $C$ in $D$ that encloses $z_0$

$$\int_C \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0)$$

the integration being taken counterclockwise.

**Lemma 2.2 (Laurent’s theorem)** Let $f(z)$ be analytic in a domain containing two concentric circles $C_1$ and $C_2$ with center $z_0$ and the annulus between them. Then $f(z)$ can be represented by the Laurent’s series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

consisting of nonnegative and negative powers. The coefficients of this Laurent’s series are given by the integral

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} \, dz^* \quad (n = 0, \pm 1, \pm 2, \cdots).$$
Theorem 2.3 Let a function $f$ be differentiable except at $x = a$. Then
\[
\int_C f(x)dx = 2\pi i \lim_{x \to a} (x - a)f(x)
\]
holds for $C$ is a simple closed path that encloses a point $a$.

Proof. Let us consider the series
\[
f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n + \frac{b_1}{x - a}.
\]
By lemma 1 and lemma 2, the coefficient $b_1$ of the $1/(x - a)$ is
\[
\frac{1}{2\pi i} \int_C f(x)dx.
\]
(1)

Multiplying $x - a$ on both sides and letting $x \to a$, from the continuity, we have
\[
\lim_{x \to a} (x - a)f(x) = b_1.
\]
(2)

By equations (1) and (2), this theorem follows.

Of course, this result is still valid in case of complex variable. Implies, we can rewrite the theorem as follows;
\[
\int_C f(z)dz = 2\pi i \lim_{z \to z_0} (z - z_0)f(z)
\]
for $C$ is a simple closed path that encloses a point $a$.

If so, let us check some examples with respect to the above theorem.

Example 2.4 Evaluate
\[
\int_0^{2\pi} \frac{d\theta}{\sqrt{2 - \cos \theta}}.
\]

Solution. Let us put $z = e^{i\theta}$. Then $\cos \theta = \frac{1}{2}(z + \frac{1}{z})$, $\sin \theta = \frac{1}{2i}(z - \frac{1}{z})$ and $d\theta = dz/iz$ hold. By theorem 2.3, the given equation equals to
\[
-\frac{i}{2} \int_C \frac{dz}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}.
\]
Organizing this equality, we have
\[
\frac{i}{2} \cdot 2\pi i \lim_{z \to \sqrt{2}-1} \frac{1}{z - \sqrt{2} + 1} = 2\pi.
\]
Example 2.5 Evaluate
\[ \int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}. \]

Solution. By the similar way with example 2.4, the given integral can be expressed by
\[ \int_C \frac{dz}{5iz - 2z^2 + 2} \]
for \( z = e^{i\theta} \). The equation (3) equals to
\[ \int_C \frac{dz}{(2z - i)(z - 2i)}. \]
Hence, by theorem 2.3, equation (3) can be rewritten as
\[
-2\pi i \lim_{z \to i/2} \left( z - \frac{i}{2} \right) \cdot \frac{1}{(2z - i)(z - 2i)} \\
= -\pi i \cdot \frac{1}{z - 2i} \bigg|_{z = i/2} = \frac{2}{3}\pi.
\]

It is clear that if a function \( f \) is not differentiable at several points such as \( a_1, a_2, \ldots, a_n \), then theorem 2.3 has to be changed to
\[ \int_C f(x)dx = 2\pi i \sum_{k=1}^n \lim_{x \to a_k} (x - a_k) f(x) \]
for \( C \) is a simple closed path that encloses points \( a_k \) and for \( (k = 1, 2 \cdots n) \).
Of course, these points can be interpreted as singular points. Let us apply this interaction formula to the Cauchy principal value. Then the Cauchy principal value of \( \int_{-\infty}^{\infty} f(x)dx \) can be obtained by
\[
2\pi \sum_{k=1}^n \lim_{x \to a_k} (x - a_k) f(x) + \pi \sum_{k=1}^n \lim_{x \to a_k} (x - a_k) f(x) \]
where the first sum extends over all singular points in the upper half-plane and the second over all singular points on the real line.

Example 2.6 Find the Cauchy principal value of
\[ \int_{-\infty}^{\infty} \frac{x + 2}{x^3 + x} \, dx. \]

Solution. The denominator \( x^3 + x \) has three singular points 0, \(-i\) and \( i \). Among these points, since \(-i\) is in the lower half-plane, there is no interest here. Thus
\[
\int_{-\infty}^{\infty} \frac{x + 2}{x^3 + x} \, dx \\
= \pi i \cdot \lim_{x \to 0} x \frac{x + 2}{x^3 + x} + 2\pi i \cdot \lim_{x \to -i} (x - i) \frac{x + 2}{x^3 + x} = \pi.
\]
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References


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