Degenerate Bernoulli Numbers and Polynomials of the Second Kind

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Abstract

In this paper, we construct degenerate Bernoulli numbers and polynomials which are slightly different Carlitz’s degenerate Bernoulli numbers and polynomials. From our degenerate Bernoulli numbers and polynomials, we derive same identities and formulae related to Bernoulli numbers and polynomials.

Keywords: Degenerate Bernoulli polynomials

1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm is normally defined by $|p|_p = \frac{1}{p}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined as
\[ I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \text{ (see [1] - [16])} \quad (1.1) \]

From (1.1), we can easily derive the following equation:

\[ I_0(f_n) - I_0(f) = \sum_{a=0}^{n-1} f'(a), \quad (n \geq 1), \quad (1.2) \]

where \( f_n(x) = f(x + 1) \), (see [4], [6])

In particular, \( n=1 \), we have

\[ I_0(f_1) - I_0(f) = f'(0) \quad (1.3) \]

From (1.3), we have

\[ \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_0(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \]

where \( B_n(x) \) are called the Bernoulli polynomials when \( x = 0 \). \( B_n \) are Bernoulli numbers. For \( \lambda, t \in \mathbb{C}_p \) with \( |\lambda t|_p < p^{\frac{1}{p-1}} \), the degenerate Bernoulli polynomials are defined by Carlitz to be

\[ \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x|\lambda) \frac{t^n}{n!}, \quad (see[1], [6]). \quad (1.4) \]

Note that \( \lim_{\lambda \to 0} \beta_n(x|\lambda) = B_n(x), \quad (n \geq 0) \).

In this paper, we consider the degenerate Bernoulli polynomials which are different Caritz’s degenerate Bernoulli polynomials. These polynomials are called the degenerate Bernoulli polynomials of the second kind. From our degenerate polynomials, we derive some interesting identities and formulae related to Bernoulli numbers and polynomials.

2. Degenerate Bernoulli polynomials of the second kind

In this section, we assume that \( \lambda, t \in \mathbb{C}_p \) such that \( |\lambda t|_p < p^{\frac{1}{p-1}} \). Now, we define the degenerate Bernoulli polynomials of the second kind as follows:

\[ \frac{1}{\lambda} \log(1 + \lambda t) \cdot (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_n(x|\lambda) \frac{t^n}{n!}, \quad (2.1) \]

From (2.1), we have
\[
\lim_{\lambda \to 0} B_n(x|\lambda) t^n/n! = \lim_{\lambda \to 0} \log \left( \frac{(1 + \lambda t)^x}{(1 + \lambda t)^x - 1} \right) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) t^n/n!.
\] (2.2)

Thus, by (2.2), we get \(\lim_{\lambda \to 0} B_n(x|\lambda) = B_n(x)\).

From (1.3), we can easily derive the following equation:

\[
\int_{\mathbb{Z}_p} (1 + \lambda t)^{x+y} d\mu_0(y) = \frac{\log (1 + \lambda t)^x}{(1 + \lambda t)^x - 1} = \sum_{n=0}^{\infty} B_n(x|\lambda) t^n/n!.
\] (2.3)

Thus, by (2.3), we get

\[
\int_{\mathbb{Z}_p} \left( \frac{x+y}{\lambda} \right)_n d\mu_0(y) = \frac{\lambda^n t^n}{n!} = \sum_{n=0}^{\infty} B_n(x|\lambda) t^n/n!.
\] (2.4)

where \((x)_n = x(x-1)\ldots(x-n+1)\).

We observe that

\[
\left( \frac{x+y}{\lambda} \right)_n = \left( \frac{x+y}{\lambda} \right) \left( \frac{x+y}{\lambda} - 1 \right) \ldots \left( \frac{x+y}{\lambda} - (n-1) \right) = \lambda^{-n} (x+y) (x+y-\lambda) \cdots (x+y-\lambda(n-1)) = \lambda^{-n} (x+y|\lambda)_n
\] (2.5)

where \((x|\lambda)_n = x(x-\lambda)(x-2\lambda)\ldots(x-\lambda(n-1))\).

From (2.4) and (2.5), we have

\[
\int_{\mathbb{Z}_p} (x+y|\lambda)_n d\mu_0(y) = B_n(x|\lambda), (n \geq 0).
\] (2.6)

On the other hand,

\[
\int_{\mathbb{Z}_p} (x+y|\lambda)_n d\mu_0(y) = \lambda^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} (x+y)^l d\mu_0(y) \lambda^{-l} = \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} B_l(x).
\] (2.7)

Thus, by (2.6) and (2.7), we get

\[
B_n(x|\lambda) = \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} B_l(x),
\] (2.8)

where \(S_1(n,l)\) is the stirling number of the first kind.
It is easy to show that
\[
\frac{\log(1 + \lambda t)^{\frac{1}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} - \frac{\log(1 + \lambda t)^{\frac{1}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} = \log(1 + \lambda t)^{\frac{1}{\lambda}}.
\]  
(2.9)

When \(x = 0\), \(B_n(\lambda) = B_n(\lambda \mid 0)\) are called the degenerate Bernoulli numbers of the second kind.

From (2.9), we have
\[
\sum_{n=0}^{\infty} B_{n+1}(1 \mid \lambda) - B_{n+1}(\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{\lambda^n(-1)^n n! t^n}{n!}.
\]  
(2.10)

Thus, by (2.10), we get
\[
B_{n+1}(1 \mid \lambda) - B_{n+1}(\lambda) = \lambda^n(-1)^n n!, \quad (n \geq 0).
\]  
(2.11)

Therefore, by (2.8) and (2.11), we obtain the following theorem.

**Theorem 2.1.** For \(n \geq 0\), we have
\[
B_0(\lambda) = B_0(1 \mid \lambda) = 1, \quad B_{n+1}(1 \mid \lambda) - B_{n+1}(\lambda) = n! \lambda^n(-1)^n,
\]
and
\[
B_n(x \mid \lambda) = \sum_{l=0}^{n} S_1(n, l) \lambda^{n-l} B_l(x).
\]

By (1.3) and (2.1), we get
\[
\sum_{n=0}^{\infty} B_n(x \mid \lambda) \frac{t^n}{n!} = \int_{\mathbb{R}} \frac{(1 + \lambda t)^{\frac{x}{\lambda}}}{(1 + \lambda t)^{\frac{x}{\lambda}} - 1} d\mu_0(y)
\]
\[
= \int_{\mathbb{R}} (1 + \lambda t)^{\frac{x}{\lambda}} d\mu_0(y) (1 + \lambda t)^{\frac{x}{\lambda}}
\]
\[
= \left( \sum_{l=0}^{\infty} B_l(\lambda) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} (x \mid \lambda) m \frac{t^m}{m!} \right)
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{\infty} B_l(\lambda) (x \mid \lambda)_{n-l} \frac{n!}{l!} \right) \frac{t^n}{n!}.
\]  
(2.12)

Thus, from (2.12), we have
\[
B_n(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} B_l(\lambda) (x \mid \lambda)_{n-l}, \quad (n \geq 0).
\]  
(2.13)
On the other hand,

\[ B_n(x \mid \lambda) = \lambda^n \int_{Z_p} \left( \frac{x + y}{\lambda} \right)_n \, d\mu_0(y) = \lambda^n n! \int_{Z_p} \left( \frac{x + y}{\lambda} \right)_n \, d\mu_0(y) \]

\[ = \lambda^n n! \sum_{l=0}^{n} \left( \frac{x}{n-l} \right) \int_{Z_p} \left( \frac{y}{l} \right) \, d\mu_0(y) \]

\[ = \lambda^n n! \sum_{l=0}^{n} \left( \frac{x}{n-l} \right) \frac{\lambda^{-l}}{l!} B_l(\lambda) = \sum_{l=0}^{n} \left( \frac{x}{n-l} \right) \frac{\lambda^{n-l} n!}{l!} B_l(\lambda) \]

\[ = \sum_{l=0}^{n} B_l(\lambda)(x \mid \lambda)_{n-l} \binom{n}{l}. \]

Note that

\[ \sum_{n=0}^{\infty} B_n(x \mid \lambda) \frac{t^n}{n!} = \left( \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \right) \left( \frac{1}{\lambda t} \log(1 + \lambda t) \right) \]

\[ = \sum_{l=0}^{\infty} \left( \frac{\beta_l(x \mid \lambda) t^l}{l!} \right) \left( \sum_{m=0}^{\infty} (-1)^m \lambda^m m! \frac{1}{m+1} t^m \right) \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m m!}{m+1} \lambda^m \beta_{n-m}(x \mid \lambda) \frac{t^n}{n!} \right) . \]

(2.14)

Thus, by (2.14), we get

\[ B_n(x \mid \lambda) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m m!}{m+1} \lambda^m \beta_{n-m}(x \mid \lambda). \]

Therefore, we obtain the following theorem.

Theorem 2.2. For \( n \geq 0 \), we have

\[ B_n(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} B_l(\lambda)(x \mid \lambda)_{n-l}, \]

and

\[ B_n(x \mid \lambda) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m m!}{m+1} \lambda^m \beta_{n-m}(x \mid \lambda). \]

Remark. By replacing \( t \) by \( \frac{1}{\lambda}(e^{\lambda t} - 1) \) in (2.3), we get
\[
\int_{Z_p} e^{(x+y)t} d\mu_0(y) = \sum_{n=0}^{\infty} B_n(x \mid \lambda) \frac{1}{n!} \lambda^{-n} (e^\lambda - 1)^n
\]
\[
= \sum_{n=0}^{\infty} B_n(x \mid \lambda) \lambda^{-n} \sum_{m=n}^{\infty} S_2(m, n) \frac{\lambda^m t^m}{m!} (2.15)
\]
\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} B_n(x \mid \lambda) \lambda^{m-n} S_2(m, n) \right) \frac{t^m}{m!}
\]

Thus, by (2.15), we get
\[
\int_{Z_p} (x+y)^m d\mu_0(y) = \sum_{n=0}^{m} B_n(x \mid \lambda) \lambda^{m-n} S_2(m, n).
\]

That is,
\[
B_m(x) = \sum_{n=0}^{m} B_n(x \mid \lambda) \lambda^{m-n} S_2(m, n), \quad (m \geq 0),
\]

where \(S_2(m, n)\) is the stirling number of the second kind.

For \(d \in \mathbb{N}\), from (1.2), we have
\[
((1 + \lambda t) \frac{d}{1} - 1) \int_{Z_p} (1 + \lambda t) \frac{d+y}{\lambda} d\mu_0(y) = \log(1 + \lambda t) \frac{d}{1} \sum_{a=0}^{d-1} (1 + \lambda t)^{\frac{a+x}{d}} (2.16)
\]

Thus, by (2.16), we get
\[
\int_{Z_p} (1 + \lambda t) \frac{d+y}{\lambda} d\mu_0(y) = \frac{\log(1 + \lambda t) \frac{d}{1} - 1}{(1 + \lambda t) \frac{d}{1}} \sum_{a=0}^{d-1} (1 + \lambda t)^{\frac{a+x}{d}}
\]
\[
= \frac{1}{d} \left( \sum_{a=0}^{d-1} \log(1 + \lambda t) \frac{d}{a} \frac{d-x}{d} (1 + \lambda t)^{\frac{d-x}{d}} \right) (2.17)
\]
\[
= \sum_{n=0}^{\infty} \left( d^{n-1} \sum_{a=0}^{d-1} B_n(\frac{a+x}{d} \mid \lambda) \frac{t^n}{n!} \right), \quad (d \in \mathbb{N}).
\]

From (2.17), we can derive the following equation :
\[
\lambda^n \int_{Z_p} \left( \frac{x+y}{\lambda} \right)_n d\mu_0(y) = d^{n-1} \sum_{a=0}^{d-1} B_n(\frac{a+x}{d} \mid \lambda), \quad (n \geq 0, \ d \in \mathbb{N}). (2.18)
\]

By, (2.4), (2.5) and (2.18), we get
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\[ B_n(x \mid \lambda) = d^{n-1} \sum_{a=0}^{d-1} B_n\left(\frac{a + x}{d} \mid \frac{\lambda}{d}\right), \quad (n \geq 0, \ d \in \mathbb{N}). \] (2.19)

Therefore, by (2.19), we obtain the following theorem.

**Theorem 2.3.** For \( n \geq 0, \ d \in \mathbb{N} \), we have

\[ B_n(x \mid \lambda) = d^{n-1} \sum_{a=0}^{d-1} B_n\left(\frac{a + x}{d} \mid \frac{\lambda}{d}\right). \]

Let \( \chi \) be a Dirichlet’s character with conductor \( d \in \mathbb{N} \). Then the generalized Bernoulli polynomials attached to \( \chi \) are defined by the generating function to be

\[ \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!} = \frac{t}{e^{dt/1}} - 1 \sum_{a=0}^{d-1} \chi(a) e^{(a+x)t}, \quad (\text{see } [15], [16], [17]). \] (2.20)

For \( d \in \mathbb{N} \), we define

\[ X = \lim_{N \to \infty} (\mathbb{Z}/dp^n\mathbb{Z}); \]
\[ a + dp^n\mathbb{Z}_p = \{ x \in X \mid x \equiv a (\text{mod } dp^n) \}; \]
\[ X^* = \bigcup_{0 < a < dp \atop p^a} (a + dp\mathbb{Z}_p). \]

We shall normally take \( 0 \leq a < dp^N \) when we write \( a + dp^N\mathbb{Z}_p \). Note that

\[ \int_X f(x) d\mu_0(x) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x), \quad (f \in UD(\mathbb{Z}_p)). \]

It is not difficult to show that

\[ \int_X \chi(y) e^{(x+y)t} d\mu_0(y) = \frac{t}{e^{dt/1}} - 1 \sum_{a=0}^{d-1} \chi(a) e^{(a+x)t} \]
\[ = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}, \quad (\text{see } [15], [16]). \] (2.21)

Thus, by (2.21), we get

\[ \int_X \chi(y)(x+y)^n d\mu_0(y) = B_{n,\chi}(x), \quad (n \geq 0). \] (2.22)

When \( x = 0 \), \( B_{n,\chi} = B_{n,\chi}(0) \) are called the generalized Bernoulli numbers attached to \( \chi \).
Now, we define the generalized degenerate Bernoulli polynomials attached to $\chi$ as follows:

$$
\sum_{n=0}^{\infty} B_{n,\chi}(x \mid \lambda) \frac{t^n}{n!} = \frac{1}{\lambda} \log(1 + \lambda t) \frac{d^{n-1}}{(1 + \lambda t)^{\frac{n}{d}} - 1} \sum_{a=0}^{d-1} \chi(a)(1 + \lambda t)^{\frac{a+x}{d}}.
$$

(2.23)

Then, by (2.23), we get

$$
\sum_{n=0}^{\infty} B_{n,\chi}(x \mid \lambda) \frac{t^n}{n!} = \frac{1}{d} \frac{d^{n-1}}{(1 + \lambda t)^{\frac{n}{d}} - 1} \sum_{a=0}^{d-1} \chi(a)(1 + \lambda t)^{\frac{a+x}{d}} \cdot \sum_{a=0}^{d-1} \chi(a) B_n \left( \frac{a+x}{d} \mid \frac{\lambda}{d} \right) \frac{t^n}{n!}.
$$

(2.24)

From (2.24), we have

$$
B_{n,\chi}(x \mid \lambda) = d^{n-1} \sum_{a=0}^{d-1} \chi(a) B_n \left( \frac{a+x}{d} \mid \frac{\lambda}{d} \right), \quad (n \geq 0).
$$

From (1.2), we have

$$
\int_X \chi(y)(1 + \lambda t)^{\frac{x+y}{\lambda}} d\mu_0(y) = \frac{1}{\lambda} \log(1 + \lambda t) \frac{d^{n-1}}{(1 + \lambda t)^{\frac{n}{d}} - 1} \sum_{a=0}^{d-1} \chi(a)(1 + \lambda t)^{\frac{a+x}{d}} \chi(a)
$$

$$
= \sum_{n=0}^{\infty} B_{n,\chi}(x \mid \lambda) \frac{t^n}{n!}.
$$

(2.25)

By (2.25), we get

$$
\int_X \chi(y)(x + y \mid \lambda) d\mu_0(y) = B_{n,\chi}(x \mid \lambda), \quad (n \geq 0).
$$

(2.26)

Therefore, we obtain the following theorem.

**Theorem 2.4.** For $n \geq 0$, $d \in \mathbb{N}$, we have

$$
B_{n,\chi}(x \mid \lambda) = d^{n-1} \sum_{a=0}^{d-1} \chi(a) B_n \left( \frac{a+x}{d} \mid \frac{\lambda}{d} \right),
$$

and

$$
\lambda^n \int_X \chi(y) \left( \frac{x+y}{\lambda} \right)^n d\mu_0(y) = \int_{\mathbb{Z}_p} \chi(y) (x + y \mid \lambda) d\mu_0(y)
$$

$$
= B_{n,\chi}(x \mid \lambda).
$$
We observe that

\[
\int_X \chi(y) \left( \frac{x+y}{\lambda} \right) d\mu_0(y) = \sum_{l=0}^{n} S_1(n, l) \int_X \chi(y) \left( \frac{x+y}{\lambda} \right)^l d\mu_0(y)
\]

\[
= \sum_{l=0}^{n} S_1(n, l) B_{l, \chi}(x) \lambda^{-l}.
\]

(2.27)

Therefore, by Theorem 2.4 and (2.27), we obtain the following corollary.

**Corollary 2.5.** For \( n \geq 0 \), we have

\[
B_{n, \chi}(x \mid \lambda) = \sum_{l=0}^{n} S_1(n, l) \lambda^{n-l} B_{l, \chi}(x).
\]

By replacing \( t \) by \( \frac{1}{\lambda}(e^{\lambda t} - 1) \) in (2.25), we get

\[
\int_X \chi(y) e^{t(x+y)} d\mu_0(y) = \sum_{n=0}^{\infty} B_{n, \chi}(x \mid \lambda) \frac{1}{n!} \lambda^{-n} (e^{\lambda t} - 1)^n
\]

\[
= \sum_{n=0}^{\infty} B_{n, \chi}(x \mid \lambda) \lambda^{-n} \sum_{m=n}^{\infty} S_2(m, n) \frac{\lambda^m}{m!} t^m
\]

\[
= \sum_{m=0}^{n} \left( \sum_{n=0}^{m} B_{n, \chi}(x \mid \lambda) S_2(m, n) \lambda^{m-n} \right) \frac{t^m}{m!}.
\]

(2.28)

Therefore, by (2.21) and (2.28), we obtain the following theorem.

**Theorem 2.6.** For \( m \geq 0 \), we have

\[
B_{m, \chi}(x) = \sum_{n=0}^{m} B_{n, \chi}(x \mid \lambda) S_2(m, n) \lambda^{m-n}.
\]

**References**


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