On the Radial Solutions of a Nonlinear Singular Elliptic Equation

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Abstract

We study the existence and asymptotic behavior near the origin of radial entire solutions of the singular elliptic equation

\[ \Delta_p U + \alpha U + \beta x \cdot \nabla U + |x|^l |U|^{q-1} U = 0 \quad \text{in } \mathbb{R}^N \]

where \( p > 2, \quad q \geq 1, \quad N \geq 1, \quad \alpha < 0, \quad \beta < 0 \) and \( l < 0 \). The behavior and the existence of positive solutions depends strongly on the sign of \( (N - p) \left( l + \frac{p(N - 1)}{p - 1} \right) \).

Mathematics Subject Classification: 35A01, 35A02, 35D05, 35K51

Keywords: Radial self-similar solutions; Singular elliptic equation; Entire solutions; Existence of positive solutions; Asymptotic behavior

1 Introduction

This paper deals with the following singular elliptic equation

\[ \Delta_p U + \alpha U + \beta x \cdot \nabla U + |x|^l |U|^{q-1} U = 0 \quad \text{in } \mathbb{R}^N \] (1.1)

where \( p > 2, \quad q \geq 1, \quad N \geq 1, \quad \alpha < 0, \quad \beta < 0 \) and \( l < 0 \). As usual, \( \nabla \) denotes the spatial gradient, while \( \Delta_p U = \text{div} (|\nabla U|^{p-2} \nabla U) \) stands for the \( p \)-Laplacian operator.
Equation (1.1) is related to the study of the following parabolic equation

\[ u_t = \Delta_p U + |x|^l |U|^{q-1} U \quad \text{in } \mathbb{R}^N \times (0, +\infty). \]  

(1.2)

In fact, equation (1.2) admits a family of radial self-similar solutions of the form

\[ v(x, t) = t^{-\alpha} u(t^{-\beta} |x|), \]  

(1.3)

defined for \( x \in \mathbb{R}^N \) and \( t > 0 \).

The scaling powers \( \alpha, \beta \) are determined by the equation in the usual manner (dimensional analysis):

\[ \alpha = \frac{l + p}{p(q - 1) + l(p - 2)}, \quad \beta = \frac{q + 1 - p}{p(q - 1) + l(p - 2)}. \]  

(1.4)

It is easy to see that \( u : \mathbb{R}^+ \rightarrow \mathbb{R} \) is a solution of the ODE

\[ \left( |u'|^{p-2} u' \right)' + \frac{N - 1}{r} |u'|^{p-2} u' + \alpha u + \beta ru' + r^l |u|^{q-1} u = 0, \quad r > 0 \]  

(1.5)

Note that for a carefully analysis of radial solutions of equation (1.1), we consider the continuous functions at the origin which do satisfy the following initial-value problem

\[ (Q) \begin{cases} 
\left( |u'|^{p-2} u' \right)' + \frac{N - 1}{r} |u'|^{p-2} u' + \alpha u + \beta ru' + r^l |u|^{q-1} u = 0, \quad r > 0, \\
u(0) = a,
\end{cases} \]

where \( p > 2, q \geq 1, N \geq 1, -p < l < 0, -N < l < 0, \alpha < 0, \beta < 0 \) and \( a \in \mathbb{R}^* \).

It’s obvious that \( u(., a, \alpha, \beta) = -u(., -a, \alpha, \beta) \), then we can restrict to the case \( a > 0 \).

The purpose of this paper is to study existence and asymptotic behavior near the origin of entire solutions of problem (Q). By an entire solution of (Q), we mean a function \( u \) defined on \([0, +\infty[\) such that \( u \in C^0([0, +\infty[) \cap C^1([0, +\infty[), |u'|^{p-2} u' \in C^1([0, +\infty[) \) and satisfying (1.5) in \([0, +\infty[\) with \( u(0) = a \).

We will show that for any solution \( u \) of problem (Q), \( \lim_{r \to 0} r^{(N-1)/(p-1)} u'(r) \) exists and is finite, hence we look for solutions of the problem

\[ (P) \begin{cases} 
\left( |u'|^{p-2} u' \right)' + \frac{N - 1}{r} |u'|^{p-2} u' + \alpha u + \beta ru' + r^l |u|^{q-1} u = 0, \quad r > 0, \\
u(0) = a, \quad \lim_{r \to 0} r^{(N-1)/(p-1)} u'(r) = b,
\end{cases} \]
Singular elliptic equation

with \( a \in \mathbb{R}^{++} \) and \( b \in \mathbb{R} \).
More precisely, we obtain the following results.
For each \( a \in \mathbb{R}^{++} \) and \( b \in \mathbb{R} \), there exists a unique solution \( u \) of problem (P) and if \( b = 0 \), we have
\[
\lim_{r \to 0} r^{-(l+1)/(p-1)} u'(r) = - \left( \frac{a^q}{l + N} \right)^{1/(p-1)}
\]
and
\[
\lim_{r \to 0} r^{(p-2-\ell)/(p-1)} u''(r) = - \frac{l + 1}{p - 1} \left( \frac{a^q}{l + N} \right)^{1/(p-1)}.
\]
On the other hand, if \( b \neq 0 \),
\[
\lim_{r \to 0} r^{(N+p-2)/(p-1)} u''(r) = - \frac{N - 1}{p - 1} b.
\]
For \( N \geq p \), necessarily \( b = 0 \). Consequently, for each \( a > 0 \), there exists a unique solution of (1.5) such that \( u(0) = a \). Moreover, if \( l > -1 \), \( u'(0) = 0 \).
So, \( u(|x|) \) is a \( C^1 \) solution of equation (1.1) in \( \mathbb{R}^N \).

Note that if \( p = 2 \) and \( l = 0 \), the equation (1.5) was studied by [6], [7], [9], [8] and [12]. If \( p = 2 \) and \( -2 < l < 0 \), it was studied by [4]. Note also that the equation was investigated in the case \( p > 2 \) and \( l = 0 \) by [2], [10] and [3] and in the case \( l < 0 \), \( \alpha > 0 \) and \( \beta > 0 \) by [5].

The rest of the paper is organized as follows.
In section 2, we study the asymptotic behavior. Section 3 concerns existence of entire solutions and under some restrictions on the initial data, we prove that any solution is strictly positive.

2 Asymptotic Behavior Near the Origin

The object of this section is to study the asymptotic behavior near 0 of solutions of the following singular equation
\[
\left( |u'|^{p-2} u' \right)' + \frac{N - 1}{r} |u'|^{p-2} u' + \alpha u + \beta r u' + r^l |u|^{q-1} u = 0, \quad r > 0, \quad (2.1)
\]
where \( p > 2 \), \( q \geq 1 \), \( N \geq 1 \), \( -p < l < 0 \), \( -N < l < 0 \), \( \alpha < 0 \) and \( \beta < 0 \).

Definition 2.1 By a solution of equation (2.1), we shall mean a function \( u \) defined on \([0, +\infty[\) such that \( u \in C^0([0, +\infty[) \cap C^1([0, +\infty[) \), \( |u'|^{p-2} u' \in C^1([0, +\infty[) \) and satisfying pointwise (2.1) in \([0, +\infty[\).

The main result is the following.
Theorem 2.2 Let \( u \) be a solution of equation (2.1) with \( u(0) = a > 0 \). Then, \( r^{(N-1)/(p-1)} u'(r) \) converges to a real constant \( b \) when \( r \) tends to 0. Moreover

- if \( b = 0 \), \( r^{-(l+1)/(p-1)} u'(r) \) converges to \( -\left(\frac{a^q}{l+N}\right)^{1/(p-1)} \) and \( r^{(p-2-l)/(p-1)} u''(r) \) converges to \( -\frac{l+1}{p-1} \left(\frac{a^q}{l+N}\right)^{1/(p-1)} \), when \( r \) tends to 0.

- if \( b \neq 0 \), \( r^{(N+p-2)/(p-1)} u''(r) \) converges to \( -\frac{N-1}{p-1} b \) when \( r \) tends to 0.

Before proving the previous theorem, we will need some preliminary results. Let us define, for all real \( c \neq 0 \), the function

\[
E_c(r) = cu(r) + ru'(r), \quad r > 0.
\]  

(2.2)

It is clear that

\[
(r^c u(r))' = r^{c-1} E_c(r), \quad r > 0,
\]  

(2.3)

therefore the monotonicity of the function \( r^c u(r) \) can be obtained by the sign of the function \( E_c(r) \).

Using equation (2.1), we have for any \( r > 0 \) such that \( u'(r) \neq 0 \),

\[
(p - 1) |u'|^{p-2} (r) E'_c(r) = (p - N + c(p - 1)) |u'|^{p-2} u'(r) - \beta r^2 u'(r) - \alpha u(r) - r^{l+1} |u|^{q-1} u(r).
\]  

(2.4)

Consequently, if \( E_c(r_0) = 0 \) for some \( r_0 > 0 \),

\[
(p - 1) |u'|^{p-2} (r_0) E'_c(r_0) = - (p - N + c(p - 1)) |c|^{p-2} c r_0^{1-p} |u|^{p-2} u(r_0) - \left[\alpha - c\beta + r_0^l |u|^{q-1}(r_0)\right] r_0 u(r_0).
\]  

(2.5)

from which we can study the sign of \( E_c(r) \).

The proof of theorem 2.2 is divided in three steps.

The first step of the proof is the following.

Proposition 2.3 Let \( u \) be a solution of equation (2.1) such that \( u(0) > 0 \), then

(i) \( u \) is strictly monotone near 0.

(ii) \( r^{(N-p)/(p-1)} u(r) \) is strictly decreasing near 0 if \( N < p \).
Proof: Assertion (i). We intend to show that \( u'(r) \neq 0 \) near 0.
Since \( u(0) > 0, l < 0 \) and \( u \) is a continuous function, we deduce that there exists a small \( R \) such that \( u(r) > 0 \) and \( r^l u^q(r) > |\alpha| \) for \( r \in ]0, R[ \).
If \( u' \) vanishes in \( ]0, R[ \) and \( r_0 \) is the first zero, we get from equation (2.1) that
\[
\left( |u'|^{p-2} u' \right)'(r_0) = - \left( \alpha + r_0^l |u|^{q-1}(r_0) \right) u(r_0) < 0.
\] (2.6)
Which in turn implies that \( u'(r) \neq 0 \) near 0.
Assertion (ii). First, observe that for \( p > N \) the function \( r^{(N-p)/(p-1)} u(r) \) cannot be increasing near 0, because \( \lim_{r \to 0} r^{(N-p)/(p-1)} u(r) = +\infty \), and it must be strictly decreasing near 0, otherwise there exists a small \( r \) such that \( E_{(N-p)/(p-1)}(r) = 0 \), then we have by (2.5)
\[
(p - 1) |u'|^{p-2} E'_{(N-p)/(p-1)}(r) = - \left( \alpha - \beta \frac{N-p}{p-1} + r^l |u|^{q-1}(r) \right) ru(r).
\] (2.7)
Since \( \lim_{r \to 0} r^l |u|^{q-1}(r) = +\infty \), then \( E'_{(N-p)/(p-1)}(r) < 0 \) and the assertion follows from (2.3).

The second step of the proof of theorem 2.2 is given by the following result.

**Proposition 2.4** Let \( u \) be a solution of equation (2.1) such that \( u(0) = a > 0 \). Then, \( \lim_{r \to 0} r^{(N-1)/(p-1)} u'(r) \) exists and finite. Moreover, if \( N \geq p \),
\[
\lim_{r \to 0} r^{(N-1)/(p-1)} u'(r) = 0.
\]

**Proof:** We introduce the following function
\[
\varphi(r) = r^{N-1} |u'|^{p-2} u'(r) + \beta r^N u(r), \quad r > 0.
\] (2.8)
According to equation (2.1), we get
\[
\varphi'(r) = r^{N-1} |u'|^{p-2} u'(r) \left[ N\beta - \alpha - r^l |u|^{q-1}(r) \right], \quad \text{for any } r > 0.
\] (2.9)
Since \( \lim_{r \to 0} r^l |u|^{q-1}(r) = +\infty \), then \( \varphi'(r) < 0 \) for small \( r \). It follows that
\[
\lim_{r \to 0} \varphi(r) \in ]-\infty, +\infty[.
\]
Since \( \lim_{r \to 0} r^N u(r) = 0 \), then by (2.8), \( \lim_{r \to 0} r^{(N-1)/(p-1)} u'(r) \in ]-\infty, +\infty[ \). We distinguish two cases:

- **\( N \geq p \).** Suppose that \( \lim_{r \to 0} r^{(N-1)/(p-1)} u'(r) \neq 0 \), then there exists a small \( R \) and a constant \( C > 0 \) such that
  \[
  |u'(r)| > Cr^{(1-N)/(p-1)}, \quad \text{for any } r \in (0, R).
  \]
  This cannot take place because \( u' \in L^1(0, R) \) and \( r^{(1-N)/(p-1)} \notin L^1(0, R) \).
  Consequently, \( \lim_{r \to 0} r^{(N-1)/(p-1)} u'(r) = 0. \)
\* \(N < p\). Suppose that \(\lim_{r \to 0} r^{(N-1)/(p-1)} u'(r) = +\infty\). The equation (2.1) can be written in the following form

\[
r^{1-N-l} (r^{N-1} |u'|^{p-2} u)' = -a^q \left[ 1 + r^{-l} |u'|^q (\alpha u + \beta ru') \right]. \tag{2.10}
\]

We know by proposition 2.3 and expression (2.3) , that \(E_{(N-p)/(p-1)}(r) < 0\) for small \(r\), that is

\[
0 < ru'(r) < -\frac{N - p}{p - 1} u(r), \quad \text{for small } r. \tag{2.11}
\]

Hence, \(ru'(r)\) is bounded near the origin. Therefore, since \(u(0) = a > 0\) and \(-l > 0\), then by (2.10), \(\lim_{r \to 0} r^{1-N-l} (r^{N-1} |u'|^{p-2} u)'(r) = -a^q\). Which implies that there exists a constant \(C > 0\) such that

\[
(r^{N-1} |u'|^{p-2} u')'(r) > -C(r^{l+N})' \quad \text{for small } r.
\]

Integrating this last inequality on \((r, r_0)\) for small \(r_0\), we get

\[
r_0^{N-1} |u'|^{p-2} u'(r_0) - r^{N-1} |u'|^{p-2} u'(r) > -Cr_0^{l+N} + Cr^{l+N}.
\]

By letting \(r \to 0\), we obtain a contradiction because the left hand side of this inequality converges to \(-\infty\). Consequently, \(\lim_{r \to 0} r^{(N-1)/(p-1)} u'(r)\) is finite. The proof is complete.

The third step of the proof of theorem 2.2 is given by

**Proposition 2.5** Let \(u\) be a solution of equation (2.1) such that \(u(0) = a > 0\).

If \(\lim_{r \to 0} r^{(N-1)/(p-1)} u'(r) = 0\), then \(r^{-(l+1)/(p-1)} u'(r)\) converges to \(-\left(\frac{a^q}{l + N}\right)^{1/(p-1)}\) when \(r\) tends to 0.

In particular, \(u'(r) < 0\) for small \(r\).

**Proof:** Since \(\lim_{r \to 0} r^{(N-1)/(p-1)} u'(r) = 0\) and \(u(0) = a\), then by (2.8), \(\lim_{r \to 0} \varphi(r) = 0\). Thus, using (2.9) and Hopital’s rule, we obtain

\[
\lim_{r \to 0} r^{-(l-N)} \varphi(r) = -\frac{a^q}{l + N}. \tag{2.12}
\]

Owing again to (2.8), we get

\[
\lim_{r \to 0} r^{-(l-1)} |u'|^{p-2} u'(r) = -\frac{a^q}{l + N}. \tag{2.13}
\]

This yields the conclusion.

We have the following important consequence,
Corollary 2.6 Let $u$ be a solution of equation (2.1) such that $u(0) > 0$. Then, $\lim_{r \to 0} ru'(r) = 0$.

Proof: the result follows easily from proposition 2.4 if $N < p$ and from proposition 2.5 if $N \geq p$.

End of the proof of theorem 2.2. From the previous propositions, we have just to give the behavior of $u''(r)$ near 0 ($u''$ exists because $u' \neq 0$ near 0). For this purpose, we distinguish two cases.

(i) If $\lim_{r \to 0} r^{(N-1)/(p-1)}u'(r) = 0$. Multiply equation (2.1) by $r^{-l}$ and use proposition 2.5 and corollary 2.6, we deduce easily

$$\lim_{r \to 0} r^{-l}\left(|u'|^{p-2}u''\right)'(r) = -\frac{a^q(l+1)}{l+N}. \quad (2.14)$$

Using again the behavior of $u'$ given by proposition 2.5, we get

$$\lim_{r \to 0} r^{(p-2-l)/(p-1)}u''(r) = -\frac{l+1}{p-1}\left(\frac{a^q}{l+N}\right)^{1/(p-1)}. \quad (2.15)$$

(ii) If $\lim_{r \to 0} r^{(N-1)/(p-1)}u'(r) = b \neq 0$. In the same way as before, multiply equation (2.1) by $r^N$ and let $r \to 0$, we obtain

$$\lim_{r \to 0} r^N\left(|u'|^{p-2}u''\right)'(r) = -(N-1)|b|^{p-2}b, \quad (2.16)$$

Hence

$$\lim_{r \to 0} r^{(N+p-2)/(p-1)}u''(r) = \frac{N-1}{p-1}b. \quad (2.17)$$

and the proof of theorem 2.2 is complete.

3 Existence of Entire Solutions

In this section, we establish the existence of entire solutions of equation (2.1). In view of the above section, if $u$ is a solution of (2.1), then necessarily $\lim_{r \to 0} r^{(N-1)/(p-1)}u'(r)$ exists and is finite. Hence, a natural problem arises:

Find a function $u$ defined on $[0, +\infty[$ such that $u \in C^0([0, +\infty[) \cap C^1([0, +\infty[)$, $|u'|^{p-2}u' \in C^1([0, +\infty[)$ and satisfying

$$(\mathcal{P}) \begin{cases} (|u'|^{p-2}u')' + \frac{N-1}{r} |u'|^{p-2}u' + \alpha u + \beta ru' + r^l|u|^{q-1}u = 0, & r > 0, \\ u(0) = a, & \lim_{r \to 0} r^{(N-1)/(p-1)}u'(r) = b, \end{cases}$$

where $\alpha$ and $\beta$ are constants.
where $p > 2$, $q \geq 1$, $N \geq 1$, $-p < l < 0$, $-N < l < 0$, $\alpha < 0$, $\beta < 0$, $a > 0$ and $b \in \mathbb{R}$.

Clearly, this problem cannot be reduced to a Cauchy problem. Hence, to establish local existence and uniqueness, we will try to convert it into a fixed point problem of some operator.

Note that the difficulty in this work lies in the fact that there was no initial data, but has only a limited condition.

**Theorem 3.1** Let $a > 0$ and $b \in \mathbb{R}$. Then, problem (P) has a unique entire solution $u$.

We shall split the proof of this theorem in two parts. The first part deals with the local existence and uniqueness, the second concerns the global existence.

**Proposition 3.2** Let $a > 0$ and $b \in \mathbb{R}$. Then, problem (P) has a unique solution $u$ defined on a maximal interval $[0, r_{\text{max}}]$.

**Proof:** Let $u$ be a solution of problem (P) on $[0, r_{\text{max}}]$. Then, for any $r \in [0, r_{\text{max}}]$, $u$ satisfies

$$
\left[ r^{N-1}|u'|^{p-2}u'(r) + \beta r^N u(r) \right]'(r) = (\beta N - \alpha) r^{N-1} u(r) - r^{l+N-1}|u|^{q-1} u(r). \tag{3.1}
$$

Since

$$
\lim_{r \to 0} r^{N-1}|u'|^{p-2}u'(r) = |b|^{p-2}b = d, \tag{3.2}
$$

then, integrating (3.1) on $(0, r)$, we obtain

$$
u(r) = a - \int_0^r G(F[u](s))ds, \tag{3.3}
$$

where

$$
G(s) = |s|^{(2-p)/(p-1)}s, \quad s \in \mathbb{R}, \tag{3.4}
$$

and the nonlinear mapping $F$ is given by

$$
F[\varphi](s) = -ds^{1-N} + \beta s \varphi(s) + s^{1-N} \int_0^s \sigma^{N-1} \left[ \alpha - \beta N + \sigma^l |\varphi|^{q-1}(\sigma) \right] \varphi(\sigma)d\sigma. \tag{3.5}
$$

Let $R > 0$, $a > M > 0$ and consider the following complete metric space:

$$
E_{a, M, R} = \{ \varphi \in C([0, R]) : \| \varphi - a \|_0 \leq M \}, \tag{3.6}
$$

where $C([0, R])$ is the Banach space of real continuous functions on $[0, R]$ with the uniform norm, denoted by $\| \cdot \|_0$.

Next we define the mapping $T$ on $E_{a, M, R}$ by

$$
T[\varphi](r) = a - \int_0^r G(F[\varphi](s))ds. \tag{3.7}
$$
The idea is to show that $T$ is a contraction from $E_{a,M,R}$ into itself for small $R$.

We will do it in two steps.

**Step 1:** $T$ maps $E_{a,M,R}$ into itself for small $M$ and $R$.

We start by limiting the function $F[\varphi]$ between two expressions which have the same sign for $\varphi \in E_{a,M,R}$.

As $\beta < 0$ and $\varphi(r) \in [a - M, a + M]$, we deduce easily that for $s \in [0, R]$,

$$F[\varphi](s) \leq -ds^{1-N} + \left[ \frac{(a + M)^q}{l + N} + \frac{|\alpha - \beta N|(a + M)}{N}s^{-l} \right]s^{l+1}. \quad (3.8)$$

If $\alpha - \beta N = 0$, we have

$$F[\varphi](s) \leq -ds^{1-N} + \frac{(a + M)^q}{l + N}s^{l+1}$$

If $\alpha - \beta N \neq 0$, we choose

$$R \leq \left( \frac{(l + N)|\alpha - \beta N|}{N(a + M)^{q-1}} \right)^{1/l}, \quad (3.9)$$

then, for any $s \in [0, R]$, we get

$$F[\varphi](s) \leq -ds^{1-N} + \frac{2(a + M)^q}{l + N}s^{l+1}. \quad (3.10)$$

Therefore, for sufficiently small $R$, we have

$$F[\varphi](s) \leq \begin{cases} 
\frac{2(a + M)^q}{l + N}s^{l+1}, & \text{if } d = 0, \\
C_1s^{1-N}, & \text{if } d \neq 0.
\end{cases} \quad (3.11)$$

where $C_1 = \frac{-d}{2} < 0$ if $d > 0$ and $C_1 = -2d > 0$ if $d < 0$.

Now, to underestimate $F[\varphi]$, we distinguish two cases.

**Case 1:** $\alpha - \beta N \geq 0$. Apply once more equation (3.5), we get

$$F[\varphi](s) \geq -ds^{1-N} + \left[ \frac{(a - M)^q}{l + N} + \beta(a + M)s^{-l} \right]s^{l+1}. \quad (3.12)$$

Since $\beta < 0$ and $l < 0$, then if we choose

$$R \leq \left( \frac{2|\beta|(l + N)(a + M)^q}{(a - M)^q} \right)^{1/l}, \quad (3.13)$$

we get for any $s \in [0, R]$

$$F[\varphi](s) \geq -ds^{1-N} + \frac{(a - M)^q}{2(l + N)}s^{l+1}. \quad (3.14)$$
Therefore, for sufficiently small $R$, we have
\[
F[\varphi](s) \geq \begin{cases} 
\frac{(a - M)^q}{2(l + N)} s^{l+1}, & \text{if } d = 0, \\
C_2 s^{l-N}, & \text{if } d \neq 0.
\end{cases}
\] (3.15)

where $C_2 = -2d < 0$ if $d > 0$ and $C_2 = -\frac{d}{2}$ if $d < 0$.

**Case 2:** $\alpha - \beta N < 0$. We see easily from (3.5) that
\[
F[\varphi](s) \geq -ds^{l-N} + \left[\frac{(a - M)^q}{l + N} + \frac{\alpha}{N}(a + M) s^{-l}\right] s^{l+1}.
\] (3.16)

Hence, as in the first case, we can choose $R$ sufficiently small such that the
(3.15) holds.

As a consequence, we obtain the following estimates
\[
\begin{cases} 
\frac{(a - M)^q}{2(l + N)} s^{l+1} \leq F[\varphi](s) \leq \frac{2(a + M)^q}{l + N} s^{l+1}, & \text{if } d = 0, \\
\frac{|d|}{2} s^{l-N} \leq |F[\varphi](s)| \leq 2|d| s^{l-N}, & \text{if } d \neq 0.
\end{cases}
\] (3.17)

Now, combining (3.7) and (3.4), we get
\[
|T[\varphi](r) - a| \leq \int_0^r |F[\varphi](s)|^{1/(p-1)} ds \quad \text{for any } r \in [0, R].
\] (3.18)

Owing to (3.17), we obtain for any $r \in [0, R]$
\[
|T[\varphi](r) - a| \leq \begin{cases} 
\frac{p - 1}{l + p} \left(\frac{2(a + M)^q}{l + N}\right)^{1/(p-1)} R^{(l+p)/(p-1)}, & \text{if } d = 0, \\
\frac{p - 1}{p - N} \left(2|d|\right)^{1/(p-1)} R^{(p-N)/(p-1)}, & \text{if } d \neq 0.
\end{cases}
\] (3.19)

So, we can choose $R$ sufficiently small such that
\[
|T[\varphi](r) - a| \leq M, \quad \text{for } \varphi \in E_{a,M,R}.
\] (3.20)

That is, $T[\varphi] \in E_{a,M,R}$.

**Step 2:** $T$ is a contraction from $E_{a,M,R}$ into itself for small $R$.

For any $r \in [0, R]$ and any $\varphi, \psi \in E_{a,M,r}$, we have
\[
|T[\varphi](r) - T[\psi](r)| \leq \int_0^r |G(F[\varphi](s)) - G(F[\psi](s))| ds
\] (3.21)

where $F[\varphi]$ is given by (3.5). Next, let
\[
\Phi(s) = \min(|F[\varphi](s)|, |F[\psi](s)|).
\]
Then
\[ |T[\varphi](r) - T[\psi](r)| \leq \int_0^r (\Phi(s))^{(2-p)/(p-1)} |F[\varphi](s) - F[\psi](s)| ds. \] (3.22)

According to (3.5) and (3.6), we have
\[ |F[\varphi](s) - F[\psi](s)| \leq \left[ \left( |\beta| + \frac{|\alpha - \beta N|}{N} \right) s + \frac{q(a + M)^{q-1}}{l + N} s^{l+1} \right] \|\varphi - \psi\|_0, \] (3.23)

As in the above we distinguish two cases:
• \( d = 0 \). Owing to (3.17), we get
\[ \Phi(s) \geq \frac{(a - M)^q}{2(l + N)} s^{l+1}. \] (3.24)

Therefore
\[ |T[\varphi](r) - T[\psi](r)| \leq \left( \frac{(a - M)^q}{2(l + N)} \right)^{2-p} \int_0^r s^{\frac{l+1}{2-p} \frac{(2-p)/p}{p-1}} |F[\varphi](s) - F[\psi](s)| ds \] (3.25)

hence, using (3.23), we get
\[ |T[\varphi](r) - T[\psi](r)| \leq \left[ K_1 r^{\frac{l(2-p)+p}{p-1}} + K_2 r^{\frac{l+q}{p-1}} \right] \|\varphi - \psi\|_0, \] (3.26)

where
\[ K_1 = \frac{p-1}{l(2-p)} + \left( |\beta| + \frac{|\alpha - \beta N|}{N} \right) \left( \frac{(a - M)^q}{2(l + N)} \right)^{(2-p)/(p-1)}, \]

and
\[ K_2 = \frac{q(p-1)}{(l + p)(l + N)} (a + M)^{q-1} \left( \frac{(a - M)^q}{2(l + N)} \right)^{(2-p)/(p-1)}, \]

• \( d \neq 0 \). Using once more (3.17), we get
\[ \Phi(s) \geq \frac{|d|}{2} s^{-1-N}. \] (3.27)

Therefore
\[ |T[\varphi](r) - T[\psi](r)| \leq \left( \frac{|d|}{2} \right)^{2-p} \int_0^r s^{\frac{l-N(2-p)}{p-1} \frac{(2-p)/p}{p-1}} |F[\varphi](s) - F[\psi](s)| ds \] (3.28)

Recall (3.23), we get
\[ |T[\varphi](r) - T[\psi](r)| \leq \left[ K_3 r^{\frac{N(p-2)+p}{p-1}} + K_4 r^{\frac{N(p-2)+p+l(p-1)}{p-1}} \right] \|\varphi - \psi\|_0, \] (3.29)
where
\[
K_3 = \frac{p-1}{N(p-2)+p} \left( |\beta| + \frac{|\alpha - \beta N|}{N} \right) \left( \frac{|d|}{2} \right)^{(2-p)/(p-1)},
\]
and
\[
K_4 = \frac{q(p-1)}{((N(p-2)+p+l(p-1))(l+N)(a+M)^q-1)} \left( \frac{|d|}{2} \right)^{(2-p)/(p-1)},
\]
Therefore, in both estimates (3.26) and (3.29), we can choose \( r \) small enough such that \( T \) is a contraction.
Consequently, the Banach Fixed Point Theorem implies the existence of unique fixed point of \( T \), which is a solution of (3.3), that is, of problem (P). The proof of proposition 3.2 is complete.

The following result concerns existence of a global solution of problem (P).

**Proposition 3.3** Let \( a > 0 \) and \( b \in \mathbb{R} \). Let \( u \) be solution of problem (P). Then, \( u \) is global.

**Proof:** Let \( u \) be a solution of problem (P) defined on \([0, r_{\text{max}}]\). We argue by contradiction assuming that \( r_{\text{max}} < +\infty \). Then
\[
\lim_{r \to r_{\text{max}}} |u(r)| = \lim_{r \to r_{\text{max}}} |u'(r)| = +\infty. \tag{3.30}
\]
We shall handle the cases \( q > 1 \) and \( q = 1 \) separately.

**Case 1:** \( q > 1 \).
For any \( r \in [0, r_{\text{max}}] \), we define the following function
\[
H_1(r) = \frac{1}{r} \left[ \frac{p-1}{p} |u'|^p(r) + \frac{\alpha}{2} u^2(r) + r^l \frac{|u|^{q+1}(r)}{q+1} \right]. \tag{3.31}
\]
According to equation (2.1), we get
\[
H_1'(r) = \frac{|u'|^p(r)}{r^2} \left[ - \left( \frac{p-1}{p} + N - 1 \right) - \beta r^2 |u'|^{2-p} + r^{l-2} |u|^{q+1}(r) \left( \frac{l-1}{q+1} - \frac{\alpha}{2} r^{-l} |u|^{-q}(r) \right) \right]. \tag{3.32}
\]
Using (3.30) and the fact that \( p > 2 \), \( N \geq 1 \), \( q > 1 \) and \( l < 0 \), we obtain
\[
\lim_{r \to r_{\text{max}}} H_1(r) = +\infty \quad \text{and} \quad \lim_{r \to r_{\text{max}}} H_1'(r) = -\infty, \tag{3.33}
\]
which is impossible.

**Case 2:** \( q = 1 \).
For any \( r \in [0, r_{\text{max}}] \), we define the following function
\[
H_2(r) = \frac{1}{r} \left[ \frac{p-1}{p} |u'|^p(r) - \frac{\alpha}{2} u^2(r) + r^l \frac{|u|^{q+1}(r)}{q+1} \right]. \tag{3.34}
\]
Again according to (3.30) and the fact that $\alpha < 0$, we have
\[ \lim_{r \to r_{\text{max}}} H_2(r) = +\infty. \] (3.35)

By direct computations, we get
\[ H'_2(r) = -\left( \frac{p-1}{p} + N - 1 \right) \frac{|u'|^p}{r^2} - \beta u'^2 + \frac{l-1}{2} r^{l-2} u'^2 - 2\alpha \frac{uu'}{r} + \frac{\alpha u^2}{2 \ r^2}. \] (3.36)

As $\alpha < 0$ and $uu' \leq |uu'| \leq \frac{p}{p+2} |u|^{(p+2)/p} + \frac{2}{p+2} |u'|^{(p+2)/2}$,
we obtain
\[ H'_2(r) \leq \frac{|u'|^p(r)}{r^2} \left[ -\left( \frac{p-1}{p} + N - 1 \right) - \beta r^2 |u'|^{2-p}(r) - \frac{4\alpha}{p+2} r |u'|^{(2-p)/2}(r) \right] \\
+ r^{l-2} u'^2(r) \left[ \frac{l-1}{2} - \frac{2\alpha p}{p+2} r^{1-l} |u|^{(2-p)/p}(r) \right]. \] (3.37)

From (3.30) and the fact that $p > 2$, $N \geq 1$ and $l < 0$, we obtain
\[ \lim_{r \to r_{\text{max}}} H'_2(r) = -\infty. \] (3.38)

But this contradicts (3.35).
Consequently, $r_{\text{max}} = +\infty$. This completes the proof.

As a consequence of proposition 2.4 and theorem 3.1, we have the following result

**Corollary 3.4** Assume $N \geq p$. Then for each $a > 0$, there exists a unique radial solution $u$ of equation (2.1) such that $u(0) = a$. Moreover,
\[ \lim_{r \to 0} r^{-(l+1)/(p-1)} u'(r) = -\left( \frac{a^q}{l+N} \right)^{1/(p-1)} \]
and
\[ \lim_{r \to 0} r^{(p-2-l)/(p-1)} u''(r) = -\frac{l+1}{p-1} \left( \frac{a^q}{l+N} \right)^{1/(p-1)}. \]

**Remark 3.5** When $N \geq p$ and $l > -1$, $u'(0) = 0$. So, $u(|x|)$ is a $C^1$ solution of equation (1.1) in $\mathbb{R}^N$.

The rest of the paper concerns the existence and uniqueness of positive solutions under some assumptions of the initial data.

For this purpose, we begin with the result which shows that any positive solution of problem (P) is strictly positive. More precisely, we have
Theorem 3.6 Let \( u \) be a solution of problem (P). If \( r_0 > 0 \) is the first zero of \( u \), then \( u'(r_0) < 0 \).

Proof: As \( r_0 > 0 \) is the first zero of \( u \), then, \( u'(r_0) \leq 0 \). Assume that \( u'(r_0) = 0 \). By continuity and the definition of \( r_0 \), there exists a left neighborhood \( (r_0 - \varepsilon, r_0) \) (for some \( \varepsilon > 0 \)) where \( u \) is strictly positive and decreasing.

We distinguish two cases.

Case 1: \( \frac{\alpha}{\beta} \leq N \).

Since \( u > 0 \) in \((0, r_0)\) and \( N\beta - \alpha \leq 0 \), then according to (2.9), \( \varphi'(r) < 0 \) in \((0, r_0)\), where the function \( \varphi \) is given by (2.8). Hence

\[
\varphi(r) > \varphi(r_0) = 0 \quad \text{for any } r \in (r_0 - \varepsilon, r_0).
\]

Which implies that

\[
u'(r) > 0 \quad \text{for any } r \in (r_0 - \varepsilon, r_0).
\]

But this contradicts the fact that \( u'(r) \leq 0 \) in \((r_0 - \varepsilon, r_0)\).

Case 2: \( \frac{\alpha}{\beta} > N \).

Define the function

\[
G(r) = r^{\alpha/\beta}|u'|^{p-2}u'(r) + \rho r^{\alpha/\beta + 1}u(r), \quad (3.39)
\]

where \( \beta < \rho < \frac{\alpha\beta}{\alpha + \beta} < 0 \).

First, note that \( G(r) < 0 \) in \((r_0 - \varepsilon, r_0)\).

On the other hand, according to equation (2.1), we have

\[
G'(r) = \left( \frac{\alpha}{\beta} - (N - 1) \right) r^{\alpha/\beta - 1}|u'|^{p-2}u'(r) + (\rho - \beta)r^{\alpha/\beta + 1}u'(r)
\]

\[
+ \left( \rho \left( \frac{\alpha}{\beta} + 1 \right) - \alpha \right) r^{\alpha/\beta} u(r) - r^{\alpha/\beta + 1}|u|^{q-1}u(r). \quad (3.40)
\]

Using the fact that \( \frac{\alpha}{\beta} > N \), \( \beta < \rho < \frac{\alpha\beta}{\alpha + \beta} < 0 \), \( u(r) > 0 \) and \( u'(r) \leq 0 \) in \((r_0 - \varepsilon, r_0)\), we have \( G'(r) < 0 \) in \((r_0 - \varepsilon, r_0)\). Hence

\[
G(r) > G(r_0) = 0 \quad \text{for any } r \in (r_0 - \varepsilon, r_0).
\]

Which is a contradiction.

Consequently, \( u'(r_0) < 0 \) and the theorem is proved.
Now, we look for solutions of the problem

\[
(P_0) \begin{cases} 
|u'|^{p-2}u' + \frac{N-1}{r}|u'|^{p-2}u' + \alpha u + \beta ru' + r^l|u|^{q-1}u = 0, & r > 0, \\
u(0) = a, \quad \lim_{r \to 0} r^{(N-1)/(p-1)} u'(r) = 0,
\end{cases}
\]

where \( p > 2, \quad q \geq 1, \quad N \geq 1, \quad -p < l < 0, \quad -N < l < 0, \quad \alpha < 0, \quad \beta < 0 \) and \( a > 0 \).

**Theorem 3.7** Assume \( N > 1 \) and \( l > -\frac{p(N-1)}{p-1} \). Then, there exists \( a_0 > 0 \) such that for any \( a \in (0, a_0) \), the solution \( u(., a) \) of problem \( (P_0) \) is strictly positive.

As the problem \( (P_0) \) is strongly singular, the perturbation method doesn’t run. So to deviate this difficulty, we use the energy method introduced by [11]; which requires this preliminary result.

**Proposition 3.8** Let \( u \) be a solution of problem \( (P_0) \). If \( r_1 \) is the first zero of \( u' \) with \( u(r_1) > 0 \), then \( (|u'|^{p-2}u')'(r_1) > 0 \).

**Proof:** We know by proposition 2.5 that \( u'(r) < 0 \) near 0, then as \( r_1 \) is the first zero of \( u' \), necessarily \( (|u'|^{p-2}u')'(r_1) \geq 0 \). Suppose that \( (|u'|^{p-2}u')'(r_1) = 0 \). So using (2.1) and the fact that \( u(r_1) > 0 \), we deduce that

\[
\alpha + r_1^l u^{q-1}(r_1) = 0.
\]

Since \( u'(r) < 0 \) in \((0, r_1)\) and \( l < 0 \), we have

\[
\alpha + r_1^l u^{q-1}(r) > \alpha + r_1^l u^{q-1}(r_1) = 0 \quad \text{for any} \quad r \in (0, r_1).
\]

Combining this inequality with (2.1), we get

\[
(r^{N-1}|u'|^{p-2}u')'(r) < 0 \quad \text{for any} \quad r \in (0, r_1).
\]

Integrating the last inequality on \((r, r_1)\) for \( r \in (0, r_1) \) and using the fact that \( u'(r_1) = 0 \), we obtain

\[
r^{N-1}|u'|^{p-2}u'(r) > 0 \quad \text{for any} \quad r \in (0, r_1),
\]

that is, \( u'(r) > 0 \) for any \( r \in (0, r_1) \), this is a contradiction. Therefore, \( (|u'|^{p-2}u')'(r_1) > 0 \) and the proof is complete.

Now, we turn to the proof of theorem 3.7.

**Proof:** (of Theorem 3.7). Let \( u \) be a solution of problem \( (P_0) \). We argue by contradiction and let \( r_0 \) the first zero of \( u \).
Indeed, since we shall show that $H$.

**Case 1:** $u'(r) < 0$ for any $r \in (0, r_0)$.

Define the function

$$H(r) = \frac{p-1}{p} |u'|^p(r) - \frac{\alpha}{2} u^2(r) - \frac{1}{q+1} r^l |u|^{q+1}(r). \tag{3.41}$$

So

$$H'(r) = -\frac{N-1}{r} |u'|^p - 2\alpha uu' - \beta ru^2 - 2r^l |u|^{q+1} uu' - \frac{l}{q+1} r^{l-1} |u|^{q+1}. \tag{3.42}$$

We claim that there exists $R \in (0, r_0)$ such that $H(R) = 0$ and $H'(R) \geq 0$. In fact, since $r_0$ is the first zero of $u$, then $u'(r_0) < 0$ by theorem 3.6. Therefore, $H(r_0) > 0$.

On the other hand, $H$ can be written in the following form

$$H(r) = r^l |u|^{q+1} \left[ \frac{-1}{q+1} - \frac{\alpha}{2} r^{-l} |u|^{-q} + \frac{p-1}{p} r^{-l} |u|^{-q-1} |u'|^p \right]. \tag{3.43}$$

This together with proposition 2.5, yields $\lim_{r \to 0} H(r) = -\frac{a^{q+1}}{q+1}$ and in turn $H(r) < 0$ for small $r$. Combining this estimate with the fact that $H(r_0) > 0$, the claim follows. Denote by $R$ the first zero of $H$.

Now, in view of (3.42), we have

$$H'(R) = \mu_1 R^{-1} u^2(R) - \frac{\mu_2}{q+1} R^{l-1} u^{q+1}(R) - 2\alpha u(R) u'(R) - \beta R u^2(R) - 2R^l u^q(R) u'(R), \tag{3.44}$$

where

$$\mu_1 = -\frac{\alpha p(N-1)}{2(p-1)} > 0 \quad \text{and} \quad \mu_2 = l + \frac{p(N-1)}{p-1} > 0.$$ 

We shall show that $H'(R) < 0$ for sufficiently small $a$.

Indeed, since $u(R) > 0$, $u'(R) < 0$ and $\alpha < 0$, then

$$H'(R) < R^{l-1} u^{q+1}(R) \left[ \frac{-\mu_2}{q+1} + \mu_1 R^{-l} u^{1-q}(R) - \beta R^{2-l} \frac{u^2(R)}{u^{q+1}(R)} + 2R \frac{|u'(R)|}{u(R)} \right]. \tag{3.45}$$

Using the fact that $H(R) = 0$, $|u'(R)|^p > 0$ and the expression (3.41) of $H(R)$, we get

$$\frac{\alpha}{2} u^2(R) + \frac{1}{q+1} R^l |u|^{q+1}(R) > 0.$$ 

Since $l < 0$ and $0 < u(R) < u(0) = a$, then

$$R = R(a) < \left( \frac{-2}{\alpha(q+1)} \right)^{-1/l} u^{-(q-1)/l}(R) < \left( \frac{-2}{\alpha(q+1)} \right)^{-1/l} a^{-(q-1)/l}. \tag{3.46}$$
Therefore, \(\lim_{a \to 0} R(a) = 0\). Finally, combining this estimate with the fact that 
\[
\lim_{r \to 0} r^{-1}u^{1-q}(r) = 0 \quad \text{and} \quad \lim_{r \to 0} ru'(r) = 0
\] 
(by corollary 2.6), we obtain 
\[
\lim_{a \to 0} R^{-1}u^{1-q}(R) = 0, 
\] (3.47)
\[
\lim_{a \to 0} \frac{R|u'(R)|}{u(R)} = 0 
\] (3.48)
and
\[
\lim_{a \to 0} R^{2-l} \frac{u^2(R)}{u^{q+1}(R)} = \lim_{a \to 0} \left( \frac{Ru'(R)}{u(R)} \right)^2 \to 0. 
\] (3.49)

In addition, if we combine those relations with (3.45) and the fact that \(\mu_2 > 0\), we find \(H'(R) < 0\) for sufficiently small \(a\) and we reach a contradiction with \(H'(R) \geq 0\).

**Case 2:** There exists \(r_1 \in (0, r_0)\) the first zero of \(u'\).
Then, \(u > 0\) in \([0, r_1]\), \(u' < 0\) in \((0, r_1)\), \(u'(r_1) = 0\) and hence by proposition 3.8, \((|u'|^{p-2}u')'(r_1) > 0\). Therefore, from equation (2.1),
\[
\left( |u'|^{p-2}u' \right)'(r_1) = -\alpha u(r_1) - r_1^q u^q(r_1) > 0. 
\] (3.50)

This together again with \(u(r_1) > 0\) and \(u'(r_1) = 0\), we get
\[
r_1^l u^{q-1}(r_1) < -\alpha 
\] (3.51)
and
\[
H(r_1) = -\frac{\alpha}{2} u^2(r_1) - \frac{1}{q+1} r_1^l u^{q+1}(r_1). 
\] (3.52)
So
\[
H(r_1) > -\frac{\alpha(q-1)}{2(q+1)} u^2(r_1) \geq 0. 
\] (3.53)

Recall that \(H(r) < 0\) for small \(r\), then there exists \(R \in (0, r_1)\) such that 
\(H(R) = 0\) and \(H'(R) \geq 0\).

The remainder of the proof is then a repetition of arguments made in the first case.
Consequently, \(u(., a)\) is strictly positive for sufficiently small \(a\). The proof is complete.

As a consequence of the previous theorem, we get the existence of positive solutions of problem (P) when \(N \geq p\).

**Theorem 3.9** Assume \(N \geq p\). Then, there exists \(a_0 > 0\) such that for any \(a \in (0, a_0)\), the solution \(u(., a)\) of problem (P) is strictly positive.
Proof: It’s easy to see that if \( N \geq p \), we have \( l > \frac{-p(N-1)}{p-1} \) (because \( l > -N \)) and by proposition 2.4, any solution \( u \) of problem (P) satisfies
\[
\lim_{r \to 0} r^{(N-1)/(p-1)} u'(r) = 0.
\]
Therefore, from theorem 3.7, \( u(.,a) \) is strictly positive for sufficiently small initial value \( a \).

References


Received: January 11, 2015; Published: April 23, 2015