Secure Connected Dominating Sets in the Join and Composition of Graphs

Amerkhan G. Cabaro

Department of Mathematics
College of Natural Sciences and Mathematics
Mindanao State University
Marawi City, Philippines

Sergio R. Canoy, Jr.

Department of Mathematics & Statistics
College of Sciences and Mathematics
Mindanao State University- Iligan Institute of Technology
9200 Iligan City, Philippines

Copyright © 2015 Amerkhan G. Cabaro and Sergio R. Canoy, Jr. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper we revisit the concept of secure connected domination in graphs. In particular, we characterized secure connected dominating sets in the join and composition of graphs and obtained the corresponding upper bounds or exact values of the secure connected domination numbers of these graphs. A rectification of a result obtained in [1] is given.

Mathematics Subject Classification: 05C69

Keywords: dominating set, connected dominating set, secure connected dominating set, external private neighbors
1 Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ of finite order and edge set $E(G)$. The neighborhood of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. If $X \subseteq V(G)$, then the open neighborhood of $X$ is the set $N_G(X) = N(X) = \bigcup \{N_G(v) : v \in X\}$. The closed neighborhood of $X$ is $N_G(X) = N[X] = X \cup N(X)$. A vertex $w \in V(G) \setminus X$ is an $X$-external private neighbor of $v \in X$ if $N_G(w) \cap X = \{v\}$. The set of all external private neighbors of $v \in X$ is denoted by $epn(v, X)$.

A subset $S$ of $V(G)$ is a dominating set (DS) in $G$ if for every $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$. $S$ is a connected dominating set (CDS) in $G$ if the induced subgraph $\langle S \rangle$ is connected. A vertex $v \in S$ is said to $S$-defend $u$, where $u \in V(G) \setminus S$ and $S$ is a CDS in $G$, if $uv \in E(G)$ and $(S \setminus \{v\}) \cup \{u\}$ is a CDS in $G$. A CDS $S$ is a secure connected dominating set (SCDS) in $G$ if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $v$ $S$-defends $u$. The domination number (resp. connected domination number and secure connected domination number) $\gamma(G)$ (resp. $\gamma_c(G)$, $\gamma_{sc}(G)$) of $G$ is the smallest cardinality of a DS (resp. CDS and SCDS) in $G$. A DS $S$ in $G$ is called a $\gamma$-set (resp. $\gamma_c$-set and $\gamma_{sc}$-set) if the cardinality of $S$ is equal to $\gamma(G)$ (resp. $\gamma_c(G)$ and $\gamma_{sc}(G)$).

The connected domination, as a variant of domination, was introduced by Sampathkumar and Walikar in [4]. The concept of secure connected domination in graphs was initially studied in [1]. This paper will revisit the latter concept of domination. Any undefined terms maybe found in [3].

2 Results and Discussion

Since every SCDS in a connected graph $G$ is a CDS in $G$, $\gamma_c(G) \leq \gamma_{sc}(G)$.

**Theorem 2.1** Let $a$ and $b$ be positive integers such that $a < b$. Then there exists a connected graph $G$ such that $\gamma_c(G) = a$ and $\gamma_{sc}(G) = b$.

**Proof:** Let $G$ be the graph shown in Figure 1. Observe that set $A = \{x_i : i = 1, 2, \ldots, a\}$ is a $\gamma_c$-set of $G$ and set $B = A \cup \{y_j : j = 1, 2, \ldots, b-a\}$ is a $\gamma_{sc}$-set of $G$. Hence, $\gamma_c(G) = |A| = a$ and $\gamma_{sc}(G) = |B| = a + (b-a) = b$.

![Figure 1: A connected graph $G$ with $\gamma_c(G) < \gamma_{sc}(G)$](image)

This proves the assertion. \hfill \qed
Corollary 2.2 The difference $\gamma_{sc} - \gamma_c$ can be made arbitrarily large.

The next result is found in [1].

Corollary 2.3 [1] Let $S$ be a CDS in $G$. Then $S$ is an SCDS in $G$ if and only if for every $u \in V(G) \setminus S$, $\exists v \in S \cap N_G(u)$ such that

(i) $epn(v, S) \subseteq N_G[u]$, and

(ii) $V(C) \cap N_G(u) \neq \emptyset$, for every component $C$ of $\langle S \setminus \{v\} \rangle$.

The following example will show that Corollary 2.3 is not the correct characterization.

Example 2.4 Let $G$ be the graph shown in Figure 2 and let $S = \{c, f\}$. Then $S$ is a CDS in $G$ and $epn(f, S) = \{e\}$. Observe that $\{e\} \subseteq N_G[d] = \{c, d, e, f\}$ and $\{e\} \cap N_G(d) = \{c\} \neq \emptyset$. Thus, conditions (i) and (ii) of Corollary 2.3 is satisfied. But $(S \setminus \{f\}) \cup \{e\} = \{c, e\}$ is not a CDS in $G$. Hence, $S$ is not an SCDS in $G$.

Before giving the correct characterization, we first give another result found in [1].

Theorem 2.5 [1] Let $S$ be a CDS in $G$, $v \in S$ and $u \in V(G) \setminus S$ with $uv \in E(G)$. Then $v$ $S$-defends $u$ if and only if $epn(v, S) \subseteq N_G(u)$ and $V(C) \cap N_G(u) \neq \emptyset$ for every component $C$ of $\langle S \setminus \{v\} \rangle$.

From Theorem 2.5, we obtain an immediate Corollary and the correct characterization.

Corollary 2.6 If $u \in epn(v, S)$ for some $v \in S$, then $u$ is not $S$-defended.

Proof: If $u \in epn(v, S)$ for some $v \in S$, then $u$ is an isolated vertex of $\langle (S \setminus \{v\}) \cup \{u\} \rangle$. It follows that $C \cap N_G(u) = \emptyset$ for every component $C$ of $S \setminus \{v\}$. Thus, by Theorem 2.5, $v$ does not $S$-defend $u$. \qed
Corollary 2.7 Let $S$ be a CDS in $G$. Then $S$ is an SCDS in $G$ if and only if

(i) $\text{epn}(v, S) = \emptyset$ for all $v \in S$, and

(ii) for every $u \in V(G) \setminus S$, $\exists v \in S \cap N_G(u)$ such that $V(C) \cap N_G(u) \neq \emptyset$ for every component $C$ of $(S \setminus \{v\})$.

Proof: Suppose $S$ is an SCDS in $G$. Then for every $u \in V(G) \setminus S$, there exists $v \in S \cap N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a CDS in $G$. This means that $v$ $S$-defends $u$. By Corollary 2.6, (i) holds. Now, by Theorem 2.5, (ii) holds. The converse follows immediately. \hfill \Box

Theorem 2.8 Let $G$ and $H$ be any graphs of orders $m \geq 2$ and $n \geq 2$, respectively. Then $S \subseteq V(G + H)$ is an SCDS in $G + H$ if and only if at least one of the following holds:

(i) $S \subseteq V(G)$ is an SCDS in $G$, where $\{z\}$ is a DS in $H$ for every $z \in V(H)$ if $|S| = 1$;

(ii) $S \subseteq V(H)$ is an SCDS in $H$, where $\{z\}$ is a DS in $G$ for every $z \in V(G)$ if $|S| = 1$;

(iii) $|S \cap V(G)| = |S \cap V(H)| = 1$ such that $S \cap V(G)$ is a DS in $G$ and $S \cap V(H)$ is a DS in $H$;

(iv) $|S \cap V(G)| = 1$ and $S \cap V(H)$ is a non-singleton DS in $H$;

(v) $|S \cap V(H)| = 1$ and $S \cap V(G)$ is a non-singleton DS in $G$;

(vi) $1 < |S \cap V(G)| \leq m$ and $1 < |S \cap V(H)| \leq n$.

Proof: Suppose $S$ is an SCDS in $G + H$. Consider the following cases:

Case 1: $S \cap V(G) = \emptyset$ or $S \cap V(H) = \emptyset$

Suppose that $S \cap V(H) = \emptyset$. Then $S \subseteq V(G)$. If $S = V(G)$, then clearly $S$ is an SCDS in $G$. Suppose $S \subset V(G)$. Let $u \in V(G) \setminus S$. Since $S$ is an SCDS in $G + H$ and $u \in V(G)$, $\exists v \in S \cap N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a CDS in $G$. Hence, $S$ is an SCDS in $G$. Now consider the case when $|S| = 1$, say $S = \{v\}$. Let $z \in V(H)$. Then $(S \setminus \{v\}) \cup \{z\} = \{z\}$ is a DS in $H$. Using the same argument, if $S \cap V(G) = \emptyset$, then $S \subseteq V(H)$ is an SCDS in $H$, where $\{z\}$ is a DS in $H$ for every $z \in V(G)$ if $|S| = 1$.

Case 2: $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$

Consider the following subcases

Subcase 1: $|S \cap V(G)| = |S \cap V(H)| = 1$

Let $\{x\} = S \cap V(G)$ and $\{y\} = S \cap V(H)$. Then $S = \{x, y\}$. Let $u \in V(G) \setminus \{x\}$. Suppose $ux \not\in E(G)$. Then $(\langle \{x\} \rangle \cup \{u\}) = \langle \{x, u\} \rangle$ is disconnected,
Subcase 2: \(|S \cap V(G)| = 1\) and \(|S \cap V(H)| \geq 2\) or \(|S \cap V(G)| \geq 2\) and \(|S \cap V(H)| = 1\).

Let \(\{x\} = S \cap V(G)\). Suppose that \(S \cap V(H)\) is not a DS in \(H\). Then \(\exists u \in V(H) \setminus (S \cap V(H))\) such that \(uv \notin E(H)\) for all \(v \in S \cap V(H)\). Thus, \(N_{G+H}(u) \cap S = \{x\}\). This implies that \(\langle (S \setminus \{x\}) \cup \{u\} \rangle\) is disconnected, contrary to the fact that \(S\) is an SCDS in \(G + H\). Hence, \(S \cap V(H)\) is a DS in \(H\). Suppose that \(S \cap V(H) = \{y\}\). Using similar argument, we can show that \(S \cap V(G)\) is a DS in \(G\).

Subcase 3: \(|S \cap V(G)| \neq 1\) and \(|S \cap V(H)| \neq 1\).

Then \(1 < |S \cap V(G)| \leq m\) and \(1 < |S \cap V(H)| \leq n\).

The converse is clear. \(\square\)

**Corollary 2.9** Let \(G\) and \(H\) be graphs of orders \(m\) and \(n\), respectively. Then \(\gamma_{sc}(G + H) = 1\) if and only if \(G = K_m\) and \(H = K_n\).

Given any subset \(C\) of \(V(G) \times V(H)\) (in fact any set of ordered pairs) can be written as \(C = \bigcup_{x \in S} \{x\} \times T_x\), where \(S \subseteq V(G)\) and \(T_x \subseteq V(H)\) for every \(x \in S\). Define the \(G\)-projection \(C_G\) as the set \(C_G = \{x \in V(G) : (x, y) \in C\text{ for some } y \in V(H)\}\). The next result is found in [2].

**Theorem 2.10** [2] Let \(G\) and \(H\) be connected graphs. Then \(C = \bigcup_{x \in S} \{x\} \times T_x\) is a \(CDS\) in \(G[H]\) if and only if \(S\) is a \(CDS\) in \(G\), where \(T_x\) is a \(CDS\) in \(H\) whenever \(|S| = 1\).

**Theorem 2.11** Let \(G\) and \(H\) be any non-trivial connected graphs. A nonempty proper subset \(C = \bigcup_{x \in S} \{x\} \times T_x\) of \(V(G[H])\) is an \(SCDS\) in \(G[H]\) if and only if \(S\) is a \(CDS\) in \(G\) satisfying the following conditions:

(i) \(T_x\) is an \(SCDS\) in \(H\) whenever \(S = \{x\}\) for some \(x \in V(G)\).

(ii) If \(|S| \geq 2\), then for each \(x \in S\) either \(T_x\) is a \(DS\) in \(H\) or \(\exists z \in N_G(x) \cap S\) such that \(|T_z| \geq 2\) or \(S \setminus \{z\}\) is a \(CDS\), where \(T_x\) is a \(CDS\) if \(|S \setminus \{z\}| = 1\).

(iii) For each \(y \in V(G) \setminus S\), \(\exists v \in N_G(y) \cap S\) such that \(|T_v| \geq 2\) or \((S \setminus \{v\}) \cup \{y\}\) is a \(CDS\) in \(G\).

**Proof:** Suppose \(C\) is an SCDS in \(G[H]\). By Theorem 2.10, \(S\) is a \(CDS\) in \(G\).

Consider the following cases:

**Case 1.** \(|S| = 1\)
Let $S = \{x\}$ and $b \in V(H) \setminus T_x$. Then $(x, b) \notin C$. Since $C$ is an SCDS, there exists $(x, a) \in C \cap N_{G[H]}(\{x, b\})$ such that $(C \setminus \{(x, a)\}) \cup \{(x, b)\}$ is a CDS in $G[H]$. This implies that $a \in T_x$ and $ab \in E(H)$. By Theorem 2.10, $(T_x \setminus \{a\}) \cup \{b\}$ is a CDS in $H$. Hence, $T_x$ is an SCDS in $H$, showing that statement (i) holds. Let $y \in V(G) \setminus S$ and suppose that $|T_x| = 1$. Pick any $c \in V(H)$. Since $C$ is an SCDS, $\exists (d, e) \in C \cap N_{G[H]}(\{y, c\})$ such that $C^* = (C \setminus \{(x, d)\}) \cup \{(y, c)\} = \{y\} \times \{c\}$ is a CDS. Thus, by Theorem 2.10, $(S \setminus \{x\}) \cup \{y\} = \{y\}$ is a CDS. This shows that (iii) holds.

**Case 2.** $|S| \neq 1$

Let $x \in S$. Since $\langle S \rangle$ is connected, $N_G(x) \cap S \neq \emptyset$. Suppose $T_x$ is not a DS in $H$. Then $\exists a \in V(H) \setminus N_H[T_x]$. Since $C$ is an SCDS, there exists $(z, b) \in C \cap N_{G[H]}((x, a))$ such that $C^* = [(C \setminus \{(z, b)\}) \cup \{(x, a)\}]$ is a CDS in $G[H]$. Since $ab \notin E(H)$, it follows that $z \in S \cap N_G(x)$. Now, since $C^* = \bigcup_{y \in S \setminus \{x, z\}} \{(y) \times T_y\} \cup \{(x) \times (T_x \cup \{a\})\} \cup \{\{z\} \times (T_x \setminus \{b\})\}$ is a CDS, the $G$-projection $S^* = C^*_G$ of $C^*$ is a CDS by Theorem 2.10. Note that $S^* = S$ or $S^* = S \setminus \{z\}$, it follows that $|T_z| \geq 2$ or $S \setminus \{z\}$ is a CDS where $T_x$ is a CDS if $|S \setminus \{x\}| = 1$. This shows that (ii) holds. Finally, let $y \in V(G) \setminus S$. Pick any $q \in V(H)$. Since $C$ is an SCDS, there exists $(v, p) \in C \cap N_{G[H]}((y, q))$ such that $C' = (C \setminus \{(v, p)\}) \cup \{(y, q)\}$ is a CDS. Hence, by Theorem 2.10, the $G$-projection $S' = C'_G$ of $C'$ is a CDS. Since $S' = S \cup \{y\}$ or $S' = (S \setminus \{v\}) \cup \{y\}$, $|T_v| \geq 2$ or $(S \setminus \{v\}) \cup \{y\}$ is a CDS. This shows that (iii) holds.

For the converse, assume that $S$ is an SCDS satisfying (i), (ii) and (iii). Then by Theorem 2.10, $C$ is a CDS. Next, let $(x, a) \notin C$. Consider the following cases:

**Case 1.** $x \in S$

Then $a \notin T_x$. If $S = \{x\}$, then by (i), $\exists d \in T_x$ such that $ad \in E(H)$ and $(T_x \setminus \{d\}) \cup \{a\}$ is a CDS in $H$. It follows that $(x, a) \in E(G[H])$ and $[(C \setminus \{(x, d)\}) \cup \{(x, a)\}]$ is a CDS in $G[H]$ by Theorem 2.10. Suppose that $|S| \neq 1$. Suppose further that $|T_z| \geq 2$ or $S \setminus \{z\}$ is a CDS, where $T_z$ is a CDS if $|S \setminus \{x\}| = 1$, for some $z \in N_G(x) \cap S$. Let $C^* = (C \setminus \{(z, b)\}) \cup \{(x, a)\}$. If $|T_z| \geq 2$, then the $G$-projection of $C^*$ is $S^* = S$. Hence, by Theorem 2.10, $C^*$ is a CDS. If $|T_z| = 1$, then $S^* = S \setminus \{z\}$ which is a CDS where $T_x$ is a CDS if $|S \setminus \{z\}| = 1$. Thus, by Theorem 2.10, $C^*$ is a CDS. Now, suppose that $T_x$ is a DS in $H$. Then there exists $d \in T_x \cap N_H(b)$. Since $S$ is a CDS and $S$ is the $G$-projection of $C^* = (C \setminus \{(x, d)\}) \cup \{(x, a)\} = \bigcup_{y \in S \setminus \{x\}} \{(y) \times T_y\} \cup \{(x) \times (T_x \setminus \{d\}) \cup \{a\}\}$, it follows from Theorem 2.10 that $C^*$ is a CDS.

**Case 2.** $x \notin S$

By (iii), choose $v \in N_G(x) \cap S$ such that $|T_v| \geq 2$ or $(S \setminus \{v\}) \cup \{x\}$ is a CDS in $G$. Choose $p \in T_v$. Then $(v, p) \in C \cap N_{G[H]}((x, a))$. Let $C' = (C \setminus \{(v, p)\}) \cup \{(x, a)\}$. Then the $G$-projection $S'$ of $C'$ is $S \cup \{x\}$ or $(S \setminus \{v\}) \cup \{x\}$. If $|T_v| \geq 2$, then $S' = S \cup \{x\}$. Note that since $S$ is a CDS and $xv \in E(G)$, $S'$ is a CDS.
Thus, by Theorem 2.10, \( C' \) is a CDS. If \( |T_v| = 1 \), then \( S' = (S \setminus \{v\}) \cup \{x\} \) is a CDS by assumption. Hence, \( C' \) is a CDS by Theorem 2.10.

Accordingly, \( C \) is an SCDS in \( G[H] \).

\[ \square \]

**Corollary 2.12** Let \( G \) and \( H \) be connected non-trivial graphs such that \( \gamma(G) \neq 1 \). Then

\[ \gamma_{sc}(G[H]) \leq \min\{\gamma_{sc}(G)\gamma(H), 2\gamma_c(G)\}. \]

**Proof:** Let \( S \) be a \( \gamma_{sc} \)-set of \( G \) and \( D \) a \( \gamma \)-set of \( H \). Since \( \gamma(G) \neq 1 \), \( |S| \geq 2 \). Set \( T_x = D \) for all \( x \in S \). Then, by Theorem 2.11, \( C = \bigcup_{x \in S} \{x\} \times T_x = S \times D \) is an SCDS in \( G[H] \). It follows that \( \gamma_{sc}(G[H]) \leq |C| = |S||D| = \gamma_{sc}(G)\gamma(H) \).

Next, let \( S \) be a \( \gamma_c \)-set of \( G \). Let \( a, b \in V(H) \), where \( a \neq b \), and set \( T_x = \{a, b\} \) for every \( x \in S \). Then, by Theorem 2.11, \( C = \bigcup_{x \in S} \{x\} \times T_x = S \times \{a, b\} \) is an SCDS in \( G[H] \). Thus, \( \gamma_{sc}(G[H]) \leq |C| = 2|S| = 2\gamma_c(G) \). Therefore,

\[ \gamma_{sc}(G[H]) \leq \min\{\gamma_{sc}(G)\gamma(H), 2\gamma_c(G)\}. \]

This proves the desired inequality. \[ \square \]

**Remark 2.13** The upper bound in Corollary 2.12 is sharp.

**Example 2.14** To illustrate Remark 2.13, let us consider the graphs shown in Figure 3. The colored vertices are their respective \( \gamma_{sc} \)-set. Thus, \( \gamma_{sc}(P_4[C_4]) = 4 = 2\gamma_c(P_4) \neq 8 = 4(2) = \gamma_{sc}(P_4)\gamma(C_4) \) and \( \gamma_{sc}(P_5[K_3]) = 5 = \gamma_{sc}(P_5)\gamma(K_3) \neq 6 = 2(3) = 2\gamma_c(P_5) \).

![Figure 3: The graphs \( P_4[C_4] \) and \( P_5[K_3] \)](image)

**Acknowledgements.** The researcher would like to thank the Commission of Higher Education of the Republic of the Philippines for the partial financial support extended through its Faculty Development Program Phase II.
References


Received: March 4, 2015; Published: April 16, 2015