Abstract

This paper is concerned with the standard finite element approximation of Hamilton-Jacobi-Bellman Equations (HJB) with nonlinear source terms. Under a realistic condition on the nonlinearity, we characterize the discrete solution as a fixed point of a contraction. As a result of this, we also derive a sharp $L^\infty$- error estimate of the approximation.

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1. INTRODUCTION

In this paper, we are interested in the finite element approximation of the Hamilton-Jacobi-Bellman equation (HJB) equation with Neuman boundary conditions:

\[
\begin{cases}
\max_{1 \leq i \leq M} (A^i u) = f(u) & \text{in } \Omega \\
\frac{\partial u}{\partial n} = 0 & \text{on } \Gamma
\end{cases}
\]  

(1.1)

where $\Omega$ is a convex bounded domain of $\mathbb{R}^N$, with smooth boundary $\Gamma$, the $A^i$s are second order uniformly elliptic operators, and $f$ is a Lipschitz nonlinearity.
HJB equations arise in many applications: stochastic control, management and economy, mechanics and optics, etc. They have been the object of intensive study during the last three decades - for a general review of their theory and applications we refer to [1],[2],[3],[4],[5],[6],[7] and the reference therein. On the numerical analysis side, and more specifically error estimates of continuous finite element approximation of HJB equations with source term independent of the solution \( u \), only few significant progresses have been made in the last fifteen years [12],[13].

Existence of a unique solution for (1.1) was discussed in [11]. In this paper we propose to study the conforming finite element approximation of this problem. For that purpose, we introduce an approach based on quasi-variational inequalities and the Banach’s fixed point Theorem. More precisely, under a realistic assumption on the nonlinearity, we characterize the solution of the corresponding discrete HJB equation as a fixed point of a contraction and, as result of this, we derive a sharp \( L^\infty \) error estimate of the approximation.

2. Assumptions and notations

We are given the second order operators

\[
\mathcal{A}^i = \sum_{1 \leq j, k \leq N} a_{jk}^i(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{k=1}^{N} b^i_k(x) \frac{\partial}{\partial x_k} + a_0^i(x)
\]

such that

\[
a_{jk}^i(x), b_k^i(x), a_0^i(x) \in C^2(\bar{\Omega}), \ x \in \bar{\Omega}, \ \forall i = 1, 2, ..., M \quad (2.1)
\]

\[
a_{jk}^i = a_{kj}^i, \ x \in \bar{\Omega}, \ \forall i = 1, 2, ..., M
\]

\[
\sum_{1 \leq j, k \leq N} a_{jk}^i(x)\xi_j \xi_k \geq \nu |\xi|^2, \nu > 0, \ \forall x \in \bar{\Omega}, \ \forall \xi \in \mathbb{R}^N, \ \forall i = 1, 2, ..., M
\]

and

\[
a_0^i(x) \geq \beta > 0, \ \forall i = 1, 2, ..., M \quad (2.2)
\]

where \( \beta \) is a positive constant. Let \((.,.)\) denote the inner product in \(L^2(\Omega)\), and

\[
a^i(u, v) = \int_{\Omega} \left( \sum_{1 \leq j, k \leq N} a_{jk}^i(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{k=1}^{N} b^i_k(x) \frac{\partial u}{\partial x_k} v + \tilde{a}_0^i(x)uv \right) dx
\]

be the bilinear forms associated with operators \( \mathcal{A}^i \), where

\[
\tilde{a}_0^i(x) = b_k^i(x) + \sum_{k=1}^{N} \frac{\partial a_{jk}^i}{\partial x_k}
\]

We assume that the bilinear forms \( a^i(.,.) \) are coercive, i.e,

\[
a^i(v, v) \geq \delta \|v\|^2_{H^1(\Omega)}, \ \delta > 0
\]
and the nonlinearity $f(.)$ is Lipschitz continuous with Lipschitz constant $c$ satisfying
\[ \frac{c}{\beta} < 1, \text{[ } \beta \text{ is defined in (2.2) } \] (2.3)

3. THE DISCRETE HJB EQUATION

We assume that $\Omega$ is polygonal. Let $\tau_h$ be a regular and quasi-uniform triangulation of $\Omega$, and denote by $h > 0$ the mesh size. Let $V_h$ denote the finite element space consisting of piecewise linear functions, $\{ \varphi_l \}; l = 1, ..., m(h)$ be the basis functions of $V_h$, and $A^i$ be the matrices with generic coefficients
\[ (A^i)_{ls} = a^i(\varphi_l, \varphi_s), \ l, s = 1, ..., m(h); \ 1 \leq i \leq M \] (3.1)

The discrete HJB equation associated with (1.1) consists of solving the following problem: find $u_h \in V_h$ solution to
\[ \max_{1 \leq i \leq M} (A^i u_h) = F(u_h) \] (3.2)

where $\quad (F(u_h))_l = (f(u_h), \varphi_l), \ l = 1, ..., m(h)$

In the sequel of the paper a discrete maximum principle (d.m.p) assumption will be needed. More precisely, we assume that the matrices $A^i$, $i = 1, 2, ..., M$, are M-Matrices [14]. Next, we shall characterize the solution of the discrete HJB equation (3.2) as the unique fixed point of a contraction.

3.1. A Contraction associated with HJB equation (3.2). Let us introduce the mapping
\[ T_h : L^\infty(\Omega) \to V_h \] (3.3)
\[ w \to T_h w = \zeta_h \]

where $\zeta_h$ is the unique solution of the following discrete HJB equation:
\[ \max_{1 \leq i \leq M} (A^i \zeta_h) = F(w) \] (3.4)

or equivalently
\[ \max_{1 \leq i \leq M} (A^i \zeta_h - F(w)) = 0 \] (3.5)

with
\[ (F(w))_l = (f(w), \varphi_l), \ l = 1, ..., m(h). \]

As $F(w)$ is independent of $\zeta_h$, thanks to [10], problem (3.4) can be approximated by the following system of QVIs: find $(\zeta^1_h, ..., \zeta^M_h) \in (V_h)^M$ such that
\[ \begin{cases} a^i(\zeta^i_h, v - \zeta^i_h) \geq (f(w), v - \zeta^i_h) \forall v \in V_h \\ \zeta^i_h \leq k + \zeta^{i+1}_h, \ v \leq k + \zeta^{i+1}_h \\ \zeta^{M+1}_h = \zeta^1_h \end{cases} \] (3.6)
**Theorem 1.** [10] Let the d.m.p hold. Then, system (3.6) has a unique solution. Moreover, as $k \to 0$, each component of the solution of this system converges uniformly in $C(\bar{\Omega})$ to the solution $\zeta_h$ of (3.4).

**Lemma 1.** Let the d.m.p hold. Then, we have

$$\max_{1 \leq i \leq M} \| \bar{\zeta}_h - \tilde{\zeta}_i \|_{\infty} \leq \frac{c}{\beta} \| w - \bar{w} \|_{\infty}, \forall w, \bar{w} \in L^\infty(\Omega)$$

**Proof.** Exactly the same as that of [[11], lemma 1].

**Theorem 2.** Under conditions of lemma 1, the mapping $T_h$ is a contraction with rate equal to $\rho = c/\beta$. Therefore $T_h$ admits a unique fixed point which coincides with the solution of HJB equation (3.2).

**Proof.** Exactly the same as that of ([11], Theorem 2).

### 3.2. $L^\infty$- Error estimate

Next, we shall derive sharp $L^\infty$ convergence order of the approximation. Also, for the rest of the paper, we will adopt $C$ as a constant independent of $h$. We begin with introducing the following auxiliary HJB equation

$$\max_{1 \leq i \leq M} (A_i \bar{\zeta}_h - F(u)) = 0 \quad (3.7)$$

where $(F(u))_l = (f(u), \varphi_l), l = 1, \ldots, m(h)$, and $u$ is the solution of the HJB equation (1.1). So, we have the following error estimate.

**Theorem 3.**

$$\| \bar{\zeta}_h - u \|_{\infty} \leq C h^2 | \log h |^2 \quad (3.8)$$

**Proof.** The proof is immediate, as $\bar{\zeta}_h$ being the discrete counterpart of $u$, making use of [13], we get (3.8).

**Theorem 4.** Let $u$ and $u_h$ be the solutions of HJB equations (1.1) and (3.2), respectively. Then

$$\| u - u_h \|_{\infty} \leq C h^2 | \log h |^2$$

**Proof.** Sine $\bar{\zeta}_h = T_h u$, and $u_h = T_h u_h$, making use of both Theorems 2 and 3, we have

$$\| u - u_h \|_{\infty} \leq \| u - \bar{\zeta}_h \|_{\infty} + \| \bar{\zeta}_h - u_h \|_{\infty} \leq \| u - \bar{\zeta}_h \|_{\infty} + \| T_h u - T_h u_h \|_{\infty} \leq C h^2 | \log h |^2 + \rho \| u - u_h \|_{\infty}$$

Thus

$$\| u - u_h \|_{\infty} \leq \frac{C h^2 | \log h |^2}{1 - \rho}$$
HJB Equations

REFERENCES


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