Prime Cycles of the Closures of the Generalized Fan
and the Generalized Wheel

Virgilio V. Britos, Jr.

Related Subjects Department
School of Engineering Technology
Mindanao State University - Iligan Institute of Technology
Andres Bonifacio Avenue, Tibanga, 9200 Iligan City, Philippines

Rosalio G. Artes, Jr. and Romulo C. Guerrero

Department of Mathematics and Statistics
College of Science and Mathematics
Mindanao State University - Iligan Institute of Technology
Andres Bonifacio Avenue, Tibanga, 9200 Iligan City, Philippines

Abstract

In this study, we established the closures of the generalized fan and the generalized wheel. Moreover, we generated results on the number of prime cycles of the resulting graphs.

Mathematics Subject Classification: 05C30

Keywords: closure of a graph, prime cycles, generalized wheel, generalized fan

1 Introduction

Let \( G \) be a connected graph of order \( n \). The closure of \( G \), denoted by \( d(G) \), is uniquely constructed from \( G \) by repeatedly adding a new edge \( uv \) connecting
a nonadjacent pair of vertices \( u \) and \( v \) with \( d_G(v) + d_G(u) \geq n \) until no more pairs with this property can be found.

An edge in a graph \( G \) joining two nonconsecutive vertices of a cycle is a chord of the cycle. A cycle in a graph \( G \) is called a prime cycle if it is chordless.

In the next section, we count the number of prime cycles of the closure of the generalized fan.

## 2 Generalized Fan: Its Closure and Prime Cycles of the Closure

The following result establishes the closure of the generalized fan.

**Theorem 2.1** \( c(\overline{K_n} + P_n) = \left\{ \begin{array}{ll} \overline{K_m} + K_n, & 1 \leq n < m \\ K_{m+n}, & 1 \leq m \leq n \leq m + 2 \\ K_m + P_n, & 1 \leq m < n - 4 \end{array} \right. \)

**Proof:** Let \( V(\overline{K_n}) = \{v_1, v_2, \ldots, v_m\} \) and \( P_n = [x_1, x_2, \ldots, x_n] \) be a partition of \( V(\overline{K_m} + P_n) \). Then \( |V(\overline{K_m} + P_n)| = m + n \). Suppose \( 1 \leq n < m \). Now, \( \deg(x_i) = m + 1 \) if \( i = 1 \) or \( i = n \) and \( \deg(x_i) = m + 2 \) if \( 1 < i < n \) and \( \deg(v_j) = n \), for all \( j \leq m \). For any vertices \( v_i, v_j \in V(\overline{K_m}) \), \( v_i v_j \notin E[c(\overline{K_m} + P_n)] \) since \( \deg(v_i) + \deg(v_j) = n + n < n + m \) where \( n < m \). Hence the resulting graph with vertices \( \{v_1, v_2, \ldots, v_m\} \) is \( \overline{K_m} \). Also, for any vertices \( x_1, x_k \in V(P_n), x_1 x_k \in E[c(\overline{K_m} + P_n)] \) if \( \deg(x_1) \geq m + 2 \) and \( \deg(x_k) \geq m + 2 \) since \( \deg(x_1) + \deg(x_k) \geq m + 2 + m + 2 \geq n + m \). Thus the resulting graph with vertices \( \{x_1, x_2, \ldots, x_n\} \) form a \( K_n \). Hence \( c(\overline{K_m} + P_n) = \overline{K_m} + K_n \).

Suppose \( 1 \leq m \leq n \leq m + 2 \). Now, \( \deg(x_i) = m + 1 \) if \( i = 1 \) or \( i = n \) and \( \deg(x_i) = m + 2 \) if \( 1 < i < n \) and \( \deg(v_j) = n \), for all \( j \leq m \). For any \( v_i, v_j \in V(\overline{K_m}), v_i v_j \in E[c(\overline{K_m} + P_n)] \) since \( \deg(v_i) + \deg(v_j) = n + n \geq n + m \) where \( n \geq m \). Hence the resulting graph with vertices \( \{v_1, v_2, \ldots, v_m\} \) form a \( K_m \). Also, for any vertices \( x_1, x_k \in V(P_n), x_1 x_k \in E[c(\overline{K_m} + P_n)] \) if \( \deg(x_1) \geq m + 2 \) and \( \deg(x_k) \geq m + 2 \) since \( \deg(x_1) + \deg(x_k) \geq m + 2 + m + 2 \geq n + m \). Thus the resulting graph with vertices \( \{x_1, x_2, \ldots, x_n\} \) form a \( K_n \). Hence \( c(\overline{K_m} + P_n) = K_m + K_n = K_{m+n} \).

Suppose \( 1 \leq m < n - 4 \). Now, \( \deg(x_i) = m + 1 \) if \( i = 1 \) or \( i = n \) and \( \deg(x_i) = m + 2 \) if \( 1 < i < n \) and \( \deg(v_j) = n \), for all \( j \leq m \). For any \( v_i, v_j \in V(\overline{K_m}), v_i v_j \in E[c(\overline{K_m} + P_n)] \) since \( \deg(v_i) + \deg(v_j) = n + n > n + m \) where \( n > m \). Hence the resulting graph with vertices \( \{v_1, v_2, \ldots, v_m\} \) form a \( K_m \). Also, for any \( x_1, x_k \in V(P_n), x_1 x_k \notin E[c(\overline{K_m} + P_n)] \) if
\[ \text{deg}(x_l) = m+2 \quad \text{and} \quad \text{deg}(x_k) = m+2 \quad \text{since} \quad \text{deg}(x_l) + \text{deg}(x_k) = m+2 + m+2 = m + m + 4 < n + m. \] Thus the resulting graph with vertices \( \{x_1, x_2, \ldots, x_n\} \) form \( P_n \). Hence \( c(K_m + P_n) = K_m + P_n \). 

The next result counts the number of prime cycles of the closure of the generalized fan.

**Theorem 2.2** For \( 3 \leq n < m \), \( c(K_m + P_n) \) has \( \left( \frac{n}{2} \right) m + \left( \frac{n}{3} \right) \) prime cycles.

**Proof:** Let \( V(K_m) = \{v_1, v_2, \ldots, v_m\} \) and \( P_n = [x_1, x_2, \ldots, x_n] \) be a partition of \( V(K_m + P_n) \) for \( 3 \leq n < m \). By definition no two vertices of \( K_m \) are adjacent with each other and \( \text{deg}(v_r) = n \), for all \( r = 1, 2, \ldots, m \). Hence \( \text{deg}(v_r) + \text{deg}(v_s) = n+n < n+m \) which is the order of \( K_m + P_n, s \leq m \). Thus, \( v_r v_s \notin E[c(K_m + P_n)] \). Note that \( x_i x_j \in E[c(K_m + P_n)], \) for all \( i, j \leq n \) since \( \text{deg}(x_i) + \text{deg}(x_j) > m + m > n + m. \) Observe now that the prime cycles in \( c(K_m + P_n) \) are triangles. Consider the following cases:

**Case 1.** Two vertices of a \( C_3 \) are from \( V(P_n) \).

These triangles are of the form \( \{v_r, x_i, x_j, v_r\} \) for \( i \neq j \) and for all \( r \leq m \). Since there are \( m \) such \( v_r \), every pair of vertices in \( V(P_n) \) forms \( m \) triangles in \( c(K_m + P_n) \). Thus, there are \( \left( \frac{n}{2} \right) m \) triangles under this case.

**Case 2.** All three vertices are from \( V(P_n) \).

Since there are \( n \) vertices in \( V(P_n) \), there are \( \left( \frac{n}{3} \right) \) triangles formed.

Hence taking the sum under these two cases, \( c(K_m + P_n) \) has \( \left( \frac{n}{2} \right) m + \left( \frac{n}{3} \right) \) prime cycles.

The number of prime cycles of the closure of the generalized wheel is established in the following result.

### 3 Generalized Wheel: Its Closure and Prime Cycles of the Closure

The closure of the generalized wheel is established in the following result.

**Theorem 3.1** \( c(K_m + C_n) = \)

\[
\begin{cases} 
K_m + K_n, & 3 \leq n < m \\
K_{m+n}, & m \leq n \leq m + 4 \\
K_m + P_n, & m \leq n - 5 
\end{cases}
\]
Proof: Let $V(K_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(C_n) = \{x_1, x_2, \ldots, x_n\}$ be a partition of $V(K_m + C_n)$. Suppose $3 \leq n < m$. Then $\deg(v_r) = n$, $r \leq m$. Thus, $\deg(v_r) + \deg(v_s) = n + n < n + m$ which is the order of $K_m + C_n$, for all $r, s \leq m$. Thus $v_r v_s \in E(c(K_m + P_n))$. Now, for all $x_i, x_j \in V(C_n)$, $\deg(x_i) + \deg(x_j) = m + 2 + m + 2 = m + m + 4 > n + m$, since $C_n$ has exactly 2 edges incident for each vertex. Hence $x_i x_j \in E(c(K_m + C_n))$, for all $i, j \leq n$ where $i \neq j$. Therefore $c(K_m + C_n) = K_m + K_n$.

Suppose $m \leq n \leq m + 4$. Then for $v_r, v_s \in V(K_m)$, $\deg(v_r) = n$ for all $r = 1, 2, \ldots, m$. Since $\deg(v_r) + \deg(v_s) = n + n \geq n + m$ which is the order of $c(K_m + C_n)$, for all $r, s \leq m$. Thus $v_r v_s \in E(c(K_m + C_n))$. Now, for all $x_i, x_j \in V(C_n)$, $\deg(x_i) + \deg(x_j) = m + 2 + m + 2 = m + m + 4 \geq n + m$. Hence $x_i x_j \in E(c(K_m + C_n))$, for all $i, j \leq n$ where $i \neq j$. Therefore $c(K_m + C_n) = K_m + C_n$.

Suppose $n \geq m + 5$. Then $\deg(v_r) = n$, for all $r = 1, 2, \ldots, m$. Since $\deg(v_r) + \deg(v_s) = n + n > n + m$ which is the order of $c(K_m + C_n)$. Now, for all $x_i, x_j \in V(C_n)$, $\deg(x_i) + \deg(x_j) = m + 2 + m + 2 = m + m + 4 < n + m$. Hence $x_i x_j \notin E(c(K_m + C_n))$, for all $i, j \leq n$ where $i \neq j$. Therefore $c(K_m + C_n) = K_m + C_n$.

The following theorem counts the number of prime cycles of the closure of the generalized wheel.

**Theorem 3.2** For $3 \leq n < m$, $c(K_m + C_n)$ has \(\binom{n}{2} m + \binom{n}{3}\) prime cycles.

Proof: Let $V(K_m) = \{v_1, v_2, \ldots, v_m\}$ and $C_n = \{x_1, x_2, \ldots, x_n\}$ be a partition of $V(K_m + C_n)$ for $3 \leq n < m$. Note that no two vertices of $K_m$ are adjacent with each other, and $\deg(v_r) = n$, for all $r = 1, 2, \ldots, m$. Hence $\deg(v_r) + \deg(v_s) = n + n < n + m$ which is the order of $K_m + C_n$, $s \leq m$. Thus, $v_r v_s \notin E(c(K_m + C_n))$. Moreover, since $\deg(x_i) + \deg(x_j) > m + m > n + m$, $x_i x_j \notin E(c(K_m + C_n))$, for all $i, j \leq n$. Observe now that the prime cycles in $c(K_m + C_n)$ are triangles $C_3$. Consider the following cases:

**Case 1.** Two vertices of a $C_3$ are from $V(C_n)$.

These triangles are of the form $\{v_r, x_i, x_j, v_r\}$ for $i \neq j$ and for all $r \leq m$. Since there are $m$ such $v_r$, every pair of vertices in $V(C_n)$ forms $m$ triangles in $c(K_m + C_n)$. Thus, there are \(\binom{n}{2} m\) triangles under this case.

**Case 2.** All three vertices are from $V(C_n)$.

Since there are $n$ vertices in $V(C_n)$, there are \(\binom{n}{3}\) triangles formed.
Hence taking the sum under the two cases, \( \overline{c(K_m + C_n)} \) has \( \binom{n}{2} m + \binom{n}{3} \) prime cycles.

References


Received: April 4, 2015; Published: May 4, 2015