Schatten Class Weighted Composition Operators on Generalized Fock Spaces $\mathcal{F}_2^2(\mathbb{C}^n)$

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Abstract

Let $\psi$ be an entire self-map of the $n$-dimensional Euclidean complex space $\mathbb{C}^n$ and $u$ be an entire function on $\mathbb{C}^n$. A weighted composition operator induced by $\psi$ with weight $u$ is given by $(uC_{\psi}f)(z) = u(z)f(\psi(z))$, for $z \in \mathbb{C}^n$ and $f$ is entire function on $\mathbb{C}^n$. In this paper, we characterize the Schatten $p$-class of weighted composition operators acting on generalized Fock space for $0 < p < \infty$.

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1. Introduction

Let $\mathbb{C}^n$ be the $n$-dimensional complex Euclidean space and let $\mathcal{H}(\mathbb{C}^n)$ be the space of all entire functions on $\mathbb{C}^n$. Suppose $\psi$ is an entire function maps $\mathbb{C}^n$ into itself and $u$ is an entire function on $\mathbb{C}^n$, the weighted composition operator $uC_{\psi}$ is defined on the space $\mathcal{H}(\mathbb{C}^n)$ of all entire functions on $\mathbb{C}^n$ by

$$(uC_{\psi}f)(z) = u(z)f(\psi(z)),$$

for all $f \in \mathcal{H}(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$. The composition operator $C_{\psi}$ is a weighted composition operator with the weight function $u$ identically equal to 1. It is
well-known that the composition operator \( C_\psi f = f \circ \psi \) defines a linear operator \( C_\psi \) which acts boundedly on spaces of entire functions on \( \mathbb{C}^n \).

Throughout this paper, we assume that \( \phi \in C^2(\mathbb{C}^n) \) is a given real valued function on \( \mathbb{C}^n \) such that

\[
mw_0 < dd^c \phi < Mw_0
\]

holds uniformly pointwise on \( \mathbb{C}^n \) for some positive constants \( m \) and \( M \) where \( d \) is the usual exterior derivative, \( d^c = \frac{i}{4}(\bar{\partial} - \partial) \), and \( w_0 = dd^c \cdot |.|^2 \) is the standard Euclidean Kähler form on \( \mathbb{C}^n \). The generalized Fock space \( F^2_\phi(\mathbb{C}^n) \) consists of all entire functions \( f \) on \( \mathbb{C}^n \) such that

\[
\| f \|_{2,\phi}^2 = \int_{\mathbb{C}^n} |f(z)|^2 e^{-2\phi(z)} dv(z)
\]

is finite, where \( dv \) is the usual Lebesgue volume measure on \( \mathbb{C}^n \). For the overview of the studies on Fock spaces we refer to the monographs [11] and [13]. If \( \phi(z) = \frac{\alpha_2}{2} |z|^2 - m \log |z| \) where \( m \) is a non-negative integer we get the Fock-Sobolev spaces \( F^{p,m} \), these spaces were first introduced by H. Cho and K. Zhu in [2].

It is well-known that, for a Hilbert space \( H \) a compact operator is in the Schatten class \( \mathcal{S}_p(H) \) if its sequence of singular numbers is in the sequence space \( \ell^p \). Recently, Ueki [12] has studied Hilbert-Schmidt of \( uC_\psi \) on the weighted Fock space \( F^2_\alpha \) for \( n = 1 \). D.Y. Du [4] has studied Schatten \( p \)-class of \( uC_\psi \) on Fock space \( F^2_\alpha(\mathbb{C}^n) \). Moreover, Mengestie [9] has studied Schatten \( p \)-class of \( uC_\psi \) on Fock-Sobolev and weighted Fock space with radial weights as well for \( p \geq 2 \). In this paper, we characterize Schatten \( p \)-class of weighted composition on the generalized Fock spaces \( F^2_\phi(\mathbb{C}^n) \). The results in this paper are generalizations of the previous ones; in fact if \( \phi(z) = \frac{\alpha}{2} |z|^2 \), for \( \alpha > 0 \), we get the results in the articles [4] and [12]. If \( \phi(z) = \frac{\alpha_2}{2} |z|^2 - m \log |z| \) where \( m \) is a non-negative integer we get the results in the paper [9].

2. Main Results

In this section we give a necessary and sufficient condition of the Schatten class membership on the generalized Fock space \( F^2_\phi \). To do this we need to introduce the definition of Toeplitz operator. It is known that the Hilbert space \( F^2_\phi \) is a closed subspace of \( L^2(\mathbb{C}^n, e^{-2\phi} dv) \) with the Bergman Kernel \( K_\phi(\cdot, \cdot) \). The Bergman orthogonal projection \( P : L^2(\mathbb{C}^n, e^{-2\phi} dv) \to F^2_\phi \) is given by integrating against a kernel \( K_\phi(w, z) = K_{\phi,z}(w) \) and by usual Hilbert space theory; that is

\[
Pf(z) = \int_{\mathbb{C}^n} K_\phi(z, w)f(w)e^{-2\phi(w)} dv(w).
\]

Recall that if \( z = (z_1, z_2, \cdots, z_n) \) and \( w = (w_1, w_2, \cdots, w_n) \) are points in \( \mathbb{C}^n \), we write \( \langle z, w \rangle = \sum_{j=1}^n z_j w_j \) and \( |z|^2 = \langle z, z \rangle \). It is well-known that the
Hilbert space $F^2_\phi$ has the Bergman reproducing kernel property that

$$f(z) = \int_{C^n} f(w) K_\phi(z, w) e^{-\phi(w)} dv(w) = \langle f, K_\phi \rangle,$$

for all $f \in F^2_\phi$ and $z \in \mathbb{C}^n$. Moreover, the inner product on $F^2_\phi$ is defined by

$$\langle f, g \rangle = \int_{C^n} f(z) \overline{g(z)} e^{-2\phi(z)} dv(z).$$

Let $k_w$ denote the normalized reproducing kernel function, which is given by $k_{\phi,w}(z) = \frac{K_\phi(z, w)}{\|K_\phi(z, w)\|_{Z,\phi}}$. For any $z \in \mathbb{C}^n$ and $r > 0$ we use

$$B(z, r) = \{w \in \mathbb{C}^n : |w - z| < r\}$$

to denote the Euclidean ball centered at $z$ with radius $r$. It is known that, see [3] for $n = 1$ and [5] for $n \geq 2$, there are positive constants $\theta$ and $C$ such that for any $z, w \in \mathbb{C}^n$

$$|K_\phi(z, w)| e^{-\phi(z)} e^{-\phi(w)} \leq C e^{-\theta |z - w|}. \quad (1)$$

Using Proposition 3.3 of [10], there are positive constants $r$, $C_1$, and $C_2$ such that for each $z \in \mathbb{C}^n$ and each $w \in B(z, r)$ we have

$$|K_\phi(z, w)| e^{-\phi(z)} e^{-\phi(w)} \geq C_1 |K_\phi(z, z)| e^{-2\phi(z)} \geq C_2. \quad (2)$$

Using (1) and (2), it is easy to see that $K_\phi(z, z) \simeq e^{2\phi(z)}$ for any $z \in \mathbb{C}^n$.

Let $\mu$ be a positive measure on $\mathbb{C}^n$, we define the Toeplitz operator $T_\mu : F^2_\phi \rightarrow F^2_\phi$ with symbol $\mu$ by

$$T_\mu f(z) = \int_{C^n} K_\phi(z, w) f(w) e^{-2\phi(w)} d\mu(w).$$

It is not clear when the above integral will converge, therefore in this paper we assume that $\mu$ is a Borel measure such that

$$\int_{C^n} e^{-\theta |z - w|} d\mu(w)$$

is finite for all $\theta > 0$ and $z \in \mathbb{C}^n$. Note that, the inequality (1) ensures that the Toeplitz operator $T_\mu$ is well-defined on the pace $F^2_\phi$. Moreover, the Berezin transformation of $\mu$ is define by $\tilde{\mu}(z) = \langle T_\mu k_{\phi,z}, k_{\phi,z} \rangle$. Using Fubini’s Theorem, it is easy to verify that

$$\tilde{\mu}(z) = \int_{C^n} |k_{\phi,z}(w)|^2 e^{-2\phi(w)} d\mu(w).$$

The following lemma gives a relation between the Toeplitz operator $T_\mu$ and the Berezin transformation of the measure $\mu$. Schatten class of Toeplitz operators on Fock spaces have been studied by the authors of [7] and [8]. This lemma is a combination of Theorem 2.7 and Theorem 3.2 in [7].

**Lemma 2.1.** Let $\mu \geq 0$ and $0 < p < \infty$. The Toeplitz operator $T_\mu$ in the Schatten class $S_p$ if and only if the Berezin transform $\tilde{\mu}$ in $L^p(\mathbb{C}^n, dv)$. 

Let $T$ be a positive operator on $F_2^2$. By mimicking the proof of Proposition 3.5 in [1] and of Theorem 6.4 in [14], it is easy to prove that $T$ is in the trace class $S_1$ if and only if $\text{tr}(T) = \int_{\mathbb{C}^n} \langle Tk_{\phi,z}, k_{\phi,z} \rangle dv(z)$ is finite. Let $\psi$ be an entire self-map of $\mathbb{C}^n$ and $u$ be an entire function of $\mathbb{C}^n$. Then we define a positive Borel measure $\mu_{u,\psi}$ on $\mathbb{C}^n$ by

$$
\mu_{u,\psi}(E) = \int_{\psi^{-1}(E)} |u(z)|^2 e^{-2\phi(z)} dv(z),
$$

where $E$ is a Borel subset of $\mathbb{C}^n$. Thus by using Theorem III.10.4 in [6], we get the following change of variable formula

$$
\int_{\mathbb{C}^n} |g(z)|^2 d\mu_{u,\psi}(z) = \int_{\mathbb{C}^n} |u(z)|^2 |g(\psi(z))|^2 e^{-2\phi(z)} dv(z),
$$

where $g$ is any entire function in the space $F_2^2(\mathbb{C}^n)$.

**Lemma 2.2.** Let $\psi$ be an entire self-map of $\mathbb{C}^n$ and let $u$ be an entire function of $\mathbb{C}^n$. If $uC_\psi : F_2^2 \rightarrow F_2^2$ is bounded, then $(uC_\psi)^*(uC_\psi) = T_{\lambda_{u,\psi}}$.

**Proof.** Suppose that $uC_\psi$ is bounded on $F_2^2$, then for any $f, g$ in $F_2^2$ we have

$$
\langle (uC_\psi)^*(uC_\psi)f, g \rangle = \langle (uC_\psi)f, (uC_\psi)g \rangle
= \int_{\mathbb{C}^n} u(z)f(\psi(z))\overline{u(z)g(\psi(z))}e^{-2\phi(z)} dv(z)
= \int_{\mathbb{C}^n} |u(z)|^2 f(\psi(z))\overline{g(\psi(z))}e^{-2\phi(z)} dv(z)
= \int_{\mathbb{C}^n} f(w)\overline{g(w)} d\mu_{u,\psi}(w)
= \int_{\mathbb{C}^n} f(w)\overline{g(w)} e^{-2\phi(w)} e^{2\phi(w)} d\mu_{u,\psi}(w).
$$
On the other hand, consider the Borel positive measure $\lambda_{u,\psi}$, where $d\lambda_{u,\psi}(z) = e^{2\phi(z)}d\mu_{u,\psi}(z)$. Then we have

$$
\langle (T_{\lambda_{u,\psi}}) f, g \rangle = \int_{\mathbb{C}^n} T_{\lambda_{u,\psi}} f(z) \overline{g(z)} e^{-2\phi(z)} dv(z)
$$

$$
= \int_{\mathbb{C}^n} \left( \int_{\mathbb{C}^n} K_{\phi,w}(z) f(w) e^{-2\phi(w)} d\lambda_{u,\psi}(w) \right) \overline{g(z)} e^{-2\phi(z)} dv(z)
$$

$$
= \int_{\mathbb{C}^n} f(w) \int_{\mathbb{C}^n} \overline{g(z)} K_{\phi,w}(z) e^{-2\phi(z)} dv(z) d\mu_{u,\psi}(w)
$$

$$
= \int_{\mathbb{C}^n} f(w) \overline{g(z)} K_{\phi,w}(z) e^{-2\phi(z)} dv(z) d\mu_{u,\psi}(w)
$$

$$
= \int_{\mathbb{C}^n} f(w) \overline{g(w)} e^{-2\phi(w)} d\lambda_{u,\psi}(w).
$$

This completes the proof. \qed

**Remark 2.3.** The Berezin transform of $\lambda_{u,\psi}$ is given by

$$
\tilde{\lambda}_{u,\psi}(z) = \int_{\mathbb{C}^n} |k_{\phi,z}(w)|^2 e^{-2\phi(w)} d\lambda_{u,\psi}(w)
$$

$$
= \int_{\mathbb{C}^n} |k_{\phi,z}(w)|^2 d\mu_{u,\psi}(w)
$$

$$
= \int_{\mathbb{C}^n} |u(w)|^2 |k_{\phi,z}(\psi(w))|^2 e^{-2\phi(w)} dv(w)
$$

$$
= \int_{\mathbb{C}^n} |u(w)|^2 |k_{\phi,z}(w)|^2 e^{-2\phi(w)} dv(w)
$$

$$
= \|uC_{\psi}k_{\phi,z}\|^2_{\phi,2}.
$$

Now we are ready to present the main result of this section.

**Theorem 2.4.** Let $0 < p < \infty$, let $\psi$ be an entire self-map of $\mathbb{C}^n$, and let $u$ be an entire function of $\mathbb{C}^n$. Suppose $uC_{\psi} : \mathcal{F}_\phi^2 \rightarrow \mathcal{F}_\phi^2$ is bounded. Then $uC_{\psi}$ is in the Schatten class $S_p$ if and only if for any $z \in \mathbb{C}^n$

$$
|u(z)|e^{\phi(z)} e^{-\phi(z)} \in L^p(\mathbb{C}^n, dv).
$$

**Proof.** Suppose $uC_{\psi}$ is bounded operator on $\mathcal{F}_\phi^2$. Then by Lemma 2.2, $(uC_{\psi})^*(uC_{\psi})$ is a Toeplitz operator with the symbol $\lambda_{u,\psi}$. It is known that, for $0 < p < \infty$ an operator $T = uC_{\psi} \in S_p$ if and only if $(T^*T)^{p/2}$ is in the trace class $S_1$, which is equivalent to say that $T^*T \in S_{p/2}$.

Hence, the composition operator $uC_{\psi}$ is bounded if and only if $T_{\lambda_{u,\psi}} \in S_{p/2}$. Using Lemma 2.1 gives that $T_{\lambda_{u,\psi}} \in S_{p/2}$ if and only if the Berezin transform $\tilde{\lambda}_{u,\psi} \in L^{p/2}(\mathbb{C}^n, dv)$. From the previous Remark 2.3, we get that

$$
\tilde{\lambda}_{u,\psi} = \|uC_{\psi}k_{\phi,z}\|^2_{\phi,2}.
$$
It is easy to see that
\[(uC_\psi)^*k_{\phi,z} = u(z)e^{-\phi(z)}k_{\phi,\psi(z)}.\]

Thus,
\[\|(uC_\psi)^*k_{\phi,z}\|_{\phi,2}^2 = |u(z)|^2e^{2\phi(\psi(z))}e^{-2\phi(z)}.\]

Therefore, \(uC_\psi \in S_p\) if and only if
\[\int_{\mathbb{C}^n} \|(uC_\psi)^*k_{\phi,z}\|_{\phi,2}^p = \int_{\mathbb{C}^n} |u(z)|^p e^{p\phi(\psi(z))}e^{-p\phi(z)} dv(z),\]
is finite. This completes the proof. \(\square\)

As an immediate consequence of the previous theorem we get the following corollary.

**Corollary 2.5.** let \(uC_\psi : \mathcal{F}_\phi^2 \rightarrow \mathcal{F}_\phi^2\) be bounded. Then \(uC_\psi\) is a Hilbert-Schmidt operator if and only if \(|u|e^{\phi(\psi)}\) is in the space \(\mathcal{F}_\phi^2\).

**REFERENCES**


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