Analytical Solutions of First-order, Periodic
Boundary Value Problem

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Abstract

In this paper, the reproducing kernel method is applied to solve first-order, periodic boundary value problems of Volterra integrodifferential equations. The analytical solution is represented in the form of convergent series with easily computable components. The solution methodology is based on generating the orthogonal basis from the obtained kernel function in the space $W_2^2[0, 1]$. The $n$-term approximation is obtained and proved to converge to the analytical solution. Numerical examples are given to demonstrate the computation efficiency of the presented method. Results obtained by the method is very simple and effective.

Mathematics Subject Classification: 35F50, 47B32, 45J05

Keywords: Reproducing kernel method, Boundary value problems, Numerical solutions

1 Introduction

Boundary value problems for differential equations play an important role in applied mathematics and physics such as elasticity, electromagnetic and fluid dynamics. However, there have been lots efforts in giving exact as well as approximate solution relating different kinds of problems in integrodifferential equations. Solving a problem means solving a set of ordinary, partial, integral
or either integrodifferential equations. Integrodifferential equations are usually
difficult to solve analytically so it is required to obtain an efficient approximate
analytical solution. Therefore, these problems have attracted considerable
attention over the last two decades [1-5, 22, 25, 32].

The purpose of this paper is to extend the application of the reproduc-
ing kernel method in the space \( W_2^2 [0, 1] \) for obtaining approximate analytical
solution for first-order, periodic boundary value problems of Volterra integrodif-
fferential equation. We consider the following equation

\[
v'(t) + \alpha v(t) = F(t, v(t), Tv(t)), \quad 0 \leq t \leq 1, \tag{1}
\]

subject to the periodic boundary condition

\[
v(0) = v(1), \tag{2}
\]

where \( \alpha \) is real finite constants, \( TV(t) = \int_0^t k(t, s)v(s) ds \), \( k(t, s) \) is continuous
function on \([0, 1] \times [0, 1] \), \( F(x, y, z) \) is continuous functions as \( y = y(t), z = z(t) \) \( \in W_2^2 [0, 1] \) and is linear or nonlinear function depending on the
problem discussed, \( v(t) \) \( \in W_2^2 [0, 1] \) is an unknown function to be determined.
We assume that Eqs. (1) and (2) has a unique smooth solution on \([0, 1] \).

Reproducing kernel theory has important application in numerical analy-
sis, differential equations, integral equations, probability and statistics, and
so on [6-9]. Recently, using the RKHS method, Geng, Abu Arqub, and oth-
ers discussed, to name but a few, two-point BVP [23, 24], fuzzy differential
equations [26], and system of equations [20, 21]. The applications of other
versions for linear and nonlinear problems can be found in [10-19, 27-31] and
references therein. The reminder of the paper is organized as follows: several
reproducing kernel spaces are described in Section 2. In Section 3, a complete
normal orthogonal system and some essential results are introduced. Also, a
method for the existence of solutions for boundary value problems (1) and (2)
based on reproducing kernel space is described. In Section 4, some numerical
examples are given to show that our method is effective and highly accurate.
The conclusions of this paper are introduced in the last section.

2 Reproducing Kernel Spaces

**Definition 2.1** Let \( E \) be a nonempty abstract set. A function \( K : E \times E \rightarrow \mathbb{C} \)
is a reproducing kernel of the Hilbert space \( H \) iff

1. \( \forall t \in E, K(\cdot, t) \in H. \)

2. \( \forall t \in E, \forall \varphi \in H, \langle \varphi, K(\cdot, t) \rangle = \varphi(t). \)
**Definition 2.2** A Hilbert spaces $H$ of functions on a set $\Omega$ is called a reproducing kernel Hilbert spaces if there exists a reproducing kernel $K$ of $H$.

**Remark 2.1** The reproducing kernel of a Hilbert space is unique, and the existence of a reproducing kernel is due to the Riesz Representation Theorem. The reproducing kernel $K$ of a Hilbert space $H$ completely determines the space $H$. Every sequence of functions $\{h_n\}_{n=1}^\infty$ which converges strongly to a function $h$ in $H$, converges also in the pointwise sense. Indeed, this convergence is uniform on every subset on $\Omega$ on which $x \to K(x, x)$ is bounded.

Next, we first construct the space $W^2_2 [0, 1]$ in which every function satisfies the boundary conditions (2) and then utilize the space $W^1_2 [0, 1]$.

### 2.1 The reproducing kernel Hilbert space $W^2_2 [0, 1]$

The inner product space $W^2_2 [0, 1]$ is defined as $W^2_2 [0, 1] = \{ v(t) \, : \, v'(t) \text{ is absolutely continuous real value function, } v''(t) \in L^2 [0, 1], \, v(0) = v(1) \}$, where $L^2 [0, 1] = \{ u \, | \, \int_0^1 v^2(t) \, dt < \infty \}$. The inner product in $W^2_2 [0, 1]$ is given by

\[
\langle u(t), v(t) \rangle_{W^2_2} = \sum_{i=0}^{1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u''(t)v''(t) \, dt, \tag{3}
\]

and the norm $\|u\|_{W^2_2}$ is denoted by $\|u\|_{W^2_2} = \sqrt{\langle u, u \rangle_{W^2_2}}$, where $u(t), v(t) \in W^2_2 [0, 1]$.

The space $W^2_2 [0, 1]$ is called a complete RKHS if for each fixed $x \in [0, 1]$ and any $u(y) \in W^2_2 [0, 1]$, there exist $K(x, y) \in W^2_2 [0, 1]$, simply $K_x(y)$, and $y \in [a, b]$ such that $\langle u(y), K_x(y) \rangle_{W^2_2} = u(x)$. Next theorem formulate the reproducing kernel $K_x(y)$.

**Theorem 2.1** The space $W^2_2[0, 1]$ is a complete reproducing kernel space, and its reproducing kernel function $R_x(y)$ can be denoted by

\[
K_x(y) = \begin{cases} 
\sum_{i=1}^{4} c_i(x)y^{i-1}, & y \leq x, \\
\sum_{i=1}^{4} d_i(x)y^{i-1}, & y > x,
\end{cases} \tag{4}
\]

**Proof.** By Eq. (3), we have

\[
\langle u(y), K_x(y) \rangle_{W^2_2} = \sum_{i=0}^{1} u^{(i)}(0)K_x^{(i)}(0) + \int_0^1 u''(y)K''(y) \, dy. \tag{5}
\]

Through several integrations by parts for Eq. (5), then

\[
\langle u(y), K_x(y) \rangle_{W^2_2} = \sum_{i=0}^{1} u^{(i)}(0) \left( K_x^{(i)}(0) + (-1)^i K_x^{(3-i)}(0) \right)
+ \sum_{i=0}^{1} (-1)^{1-i} u^{(i)}(1) K_x^{(3-i)}(1) + \int_0^1 u(y) K_x^{(4)}(y) \, dy.
\]
Since $K_x(y) \in W^2_2[0, 1]$, it follows that
\[ K_x(0) = K_x(1), \]
and since $u(y) \in W^2_2[0, 1]$, $u(0) = u(1)$. Hence, we have
\[
\langle u(y), R_x(y) \rangle_{W^2_2} = \sum_{i=0}^{1} u^{(i)}(0) \left[ R^{(i)}_x(0) + (-1)^i R^{(3-i)}_x(0) \right]
+ \sum_{i=0}^{1} (-1)^{1-i} u^{(i)}(1) R^{(3-i)}_x(1)
+ \int_0^1 u(y) R^{(4)}_x(y) \, dy + c_1(u(0) - u(1)).
\tag{7}
\]

If $K''_x(1) = 0, K'_x(0) + K^{(3)}_x(0) + c_1 = 0, K'_x(0) - K^{(2)}_x(0) = 0, K^{(3)}_x(1) + c_1 = 0$, then Eq. (7) implies that
\[
\langle u(y), K_x(y) \rangle_{W^2_2} = \int_0^1 u(y) K^{(4)}_x(y) \, dy.
\]
For $\forall y \in [0, 1]$, if $K_x(y)$ also satisfies
\[
K^{(4)}_x(y) = \delta(y - x), \tag{9}
\]
then $\langle u(y), K_x(y) \rangle_{W^2_2} = u(x)$. The characteristic equation of Eq. (9) is given by $\lambda^4 = 0$, then we can obtain the characteristic values $\lambda = 0$ (a 4 multiple roots). Hence, $K_x(y)$ can be written as in Eq. (4).

On the other hand, for Eq. (9), let $K_x(y)$ satisfies
\[
K^{(m)}_x(x + 0) = K^{(m)}_x(x - 0), m = 0, 1, 2. \tag{10}
\]
Integrating Eq. (9) from $x - \varepsilon$ to $x + \varepsilon$ with respect to $y$ and let $\varepsilon \to 0$, we have the jump degree of $K^{(3)}_x(y)$ at $y = x$ given by
\[
K^{(3)}_x(x - 0) - K^{(3)}_x(x + 0) = -1. \tag{11}
\]
Applying Eqs. (6), (8), (10), (11), the unknown coefficient of Eq. (4) can be obtained by using Mathematica software package such that
\[
c_1(x) = 1, c_2(x) = \frac{1}{8} x(2 - 3x + x^2),
c_3(x) = \frac{1}{16} x(2 - 3x + x^2), c_4(x) = \frac{1}{48} (-8 + 6x + 3x^2 - x^3),
d_1(x) = 1 - \frac{1}{6} x^3, d_2(x) = \frac{1}{8} x(2 + x + x^2),
d_3(x) = -\frac{1}{36} x(6 + 3x - x^2), d_4(x) = \frac{1}{48} x(6 + 3x + x^2).
\]
Hence, the reproducing kernel function $K_x(y)$ is given by

$$K_x(y) = \begin{cases} \frac{1}{48}[x^3y(6 + 3y - y^2) + 3x^2y(-6 - 3yy^2) + 6xy(2 + y + y^2) - 8(-6 + y^3)], & y \leq x, \\ \frac{1}{48}[48 + 6xy(2 - 3y + y^2) + 3x^2y(2 - 3y + y^2) - x^3(8 - 6y - 3y^2 + y^3)], & y > x. \end{cases}$$

**Corollary 2.1** The reproducing kernel $K_x(y)$ is symmetric, unique, and $K_x(x) \geq 0$ for any fixed $x \in [0, 1]$.

**Proof.** By the reproducing property, we have $K_x(y) = \langle K_x(\xi), K_y(\xi) \rangle = \langle K_y(\xi), K_x(\xi) \rangle = K_y(x)$ for each $x$ and $y$. Now, let $K^1_x(y)$ and $K^2_x(y)$ be all the reproducing kernels of the space $W^2_2[0, 1]$, then $K^1_x(y) = \langle K^1_x(\xi), K^2_y(\xi) \rangle = \langle K^2_y(\xi), K^1_x(\xi) \rangle = K^2_y(x) = K^2_x(y)$.

Finally, we note that $K_x(x) = \langle K_x(\xi), K_x(\xi) \rangle = \|K_x(\xi)\|^2 \geq 0$.

### 2.2 The reproducing kernel Hilbert space $W^2_2[0, 1]

The inner product space $W^2_2[0, 1]$ is defined as $W^2_2[0, 1] = \{u(t) : u(t)$ is absolutely continuous real valued function on $[0, 1]$ and $u' \in L^2[0, 1]\}$. The inner product in $W^2_2[0, 1]$ is given by

$$\langle u, v \rangle_{W^2_2} = \int_0^1 (u'(t)v'(t) + u(t)v(t)) \, dt,$$

and the norm $\|u\|_{W^2_2}$ is denoted by $\|u\|_{W^2_2} = \sqrt{\langle u, u \rangle_{W^2_2}}$, where $u, v \in W^2_2[0, 1]$.

In [8], the authors have proved that the space $W^2_2[0, 1]$ is a RKHS and its reproducing kernel is

$$R_x(y) = \cosh(x + y - 1) + \cosh(|x + y| - 1)) / 2 \sinh(1).$$

From the definitions of the reproducing kernel spaces $W^2_2[0, 1]$ and $W^1_2[0, 1]$, clearly that $W^1_2[0, 1] \supset W^2_2[0, 1]$ for any $u(x) \in W^2_2[0, 1]$ and $\|u\|_{W^1_2} \leq \|u\|_{W^2_2}$.

### 3.2 Introduction into a linear operator

Define a differential operator $L : W^2_2[0, 1] \to W^1_2[0, 1]$ such that $Lu(x) = v'(x) + \alpha v(x)$. After homogenization of the initial conditions, then boundary value problems (1) and (2) can be converted into the following form

$$Lu(x) = F(x, u(x), Tu(x)), \ x \in (0, 1);$$

$$u(0) = u(1), \quad (12)$$

where $Tu(x) = \int_0^x h(x, s)u(s)ds$, $u(x) \in W^2_2[0, 1]$ and $F(x, y, z) \in W^1_2[0, 1]$ as $y = y(x), z = z(x) \in W^2_2[0, 1]$, for $y, z \in (-\infty, \infty)$, and $x \in [0, 1]$. It is clear that $L$ is a bounded linear operator.
Lemma 2.1 If \( u(x) \in W^2_2[0,1] \), then \( \|u^{(i)}(x)\|_{L^\infty} \leq M_i \|u(x)\|_{W^2_2} \), where \( M_i, i = 0, 1 \), are positive constants.

Proof. For any \( x \in [0,1] \), it holds that \( \|K_x(y)\|_{W^2_2} = \sqrt{\langle K_x(y), K_x(y) \rangle_{W^2_2}} = \sqrt{K_x(x)} \). From the continuity of \( K_x(x) \), there exists a constant \( M_0 \), such that \( \|K_x(y)\|_{W^2_2} \leq M_0 \). By the expression of \( K_x(y) \), one gets

\[
|u(x)| = |\langle u(x), K_x(x) \rangle_{W^2_2}| \leq \|K_x(x)\|_{W^2_2} \|u(x)\|_{W^2_2} \leq M_0 \|u(x)\|_{W^2_2} .
\]

Similarly, for any \( x, y \in [0,1] \), there exists a constant \( M_1 \), such that \( \|K_x'(y)\|_{W^2_2} \leq M_1 \). Therefore, \( |u'(x)| \leq M_1 \|u(x)\|_{W^2_2} \). Hence, \( \|u^{(i)}(x)\|_{L^\infty} \leq M_i \|u(x)\|_{W^2_2} \)

(i = 0, 1).

3  An Orthogonal Basis

Now, we construct an orthogonal function system of \( W^2_2[0,1] \). For a fixed dense set \( \{x_i\}_{i=1}^\infty \) of \([0,1]\), let \( \varphi_i(x) = R_{x_i}(x) \). So, from the properties of \( R_{x_i}(y) \), for every \( u(x) \in W^2_2[0,1] \), it follows that \( \langle u(x), \varphi_i(x) \rangle_{W^2_2} = \langle u(x), R_{x_i}(x) \rangle_{W^2_2} = u(x_i) \). Additionally, let \( \psi_i(x) = L^*\varphi_i(x) \), where \( L^* \) is the adjoint operator of \( L \). Obviously, \( \psi_i(x) \in W^2_2[0,1] \). In terms of the properties of \( K_x(y) \), one obtains \( \langle u(x), \psi_i(x) \rangle_{W^2_2} = \langle u(x), L^*\varphi_i(x) \rangle_{W^2_2} = \langle Lu(x), \varphi_i(x) \rangle_{W^2_2} = Lu(x_i), i = 1, 2, ... \).

Lemma 3.1 \( \varphi_i(x) \) can be expressed in the form \( \psi_i(x) = LgK_x(y)|_{y=x_i} \). The subscript \( y \) by the operator \( L \) indicates that the operator \( L \) applies to the function of \( y \).

Proof. From the above assumption, it is clear that \( \psi_i(x) = L^*\varphi_i(x) = \langle L^*\varphi_i(x), K_x(y) \rangle_{W^2_2} = \langle \varphi_i(x), LgK_x(y) \rangle_{W^2_2} = LgK_x(y)|_{y=x_i} \).

Theorem 3.1 If \( \{x_i\}_{i=1}^\infty \) is dense on \([0,1]\), then \( \{\psi_i(x)\}_{i=1}^\infty \) is the complete function system of \( W^2_2[0,1] \).

Proof. For each fixed \( u(x) \in W^2_2[0,1] \), let \( \langle u(x), \psi_i(x) \rangle = 0, i = 1, 2, ... \), that is \( \langle u(x), \psi_i(x) \rangle = \langle u(x), L^*\varphi_i(x) \rangle = \langle Lu(x), \varphi_i(x) \rangle = Lu(x_i) = 0 \). Note that \( \{x_i\}_{i=1}^\infty \) is dense on \([0,1]\), therefore, \( Lu(x) = 0 \). It follows that \( u(x) = 0 \) from the existence of \( L^{-1} \). So, the proof of the Theorem is complete.

The orthonormal function system \( \{\overline{\psi}_i(x)\}_{i=1}^\infty \) of \( W^2_2[0,1] \) can be derived from Gram-Schmidt orthogonalization process of \( \{\psi_i(x)\}_{i=1}^\infty \) as follows:

\[
\overline{\psi}_i(x) = \sum_{k=1}^i \beta_{ik} \psi_k(x),
\]

where \( \beta_{ik} \) are orthogonalization coefficients given as:

\[
\beta_{11} = 1/\|\psi_1\|, \quad \beta_{ii} = 1/\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}, i = 1, 2, ..., \quad \text{and} \quad \beta_{ij} = -\sum_{k=j}^{i-1} c_{ik} \beta_{kj} \sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2} \quad \text{for} \ j < i \text{ in which} \ c_{ik} = \langle \psi_i, \overline{\psi}_k \rangle_{W^2_2} .
\]
The structure of the next two theorems are as follows: Firstly, we will give the representation of the exact solution of boundary value problems (1) and (2) in the space $W^2_2[0,1]$. After that, the convergence of approximate solution $u_n(x)$ to the analytic solution will be proved.

**Theorem 3.2** For each $u(x) \in W^2_2[0,1]$, the series $\sum_{i=1}^{\infty} \langle u(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x)$ is convergent in the norm of $W^2_2[0,1]$. On the other hand, if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and the solution of the boundary value problems (1) and (2) is unique, then this solution satisfies the form:

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k)) \bar{\psi}_i(x). \quad(14)$$

**Proof.** Applying Theorem 3.1, it is easy to see that $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W^2_2[0,1]$. Thus, $u(x)$ can be expanded in the Fourier series as $u(x) = \sum_{i=1}^{\infty} \langle u(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x)$ about normal orthogonal system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$. Moreover, $W^2_2[0,1]$ is a Hilbert space, then the series $\sum_{i=1}^{\infty} \langle u(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x)$ is convergent in the norm of $W^2_2[0,1]$.

Since $\langle v(x), \varphi_i(x) \rangle = v(x_i)$ for each $v(x) \in W^1_2[0,1]$. Hence, we have

$$u(x) = \sum_{i=1}^{\infty} \langle u(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), \psi_k(x) \rangle \bar{\psi}_i(x)
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), L^* \varphi_k(x) \rangle \bar{\psi}_i(x)
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle L \psi(x), \varphi_k(x) \rangle \bar{\psi}_i(x)
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k)) \bar{\psi}_i(x).$$

**Remark 3.1** If Equation (1) is linear, then the analytical solution can be obtained directly from Equation (14). In the case of Equation (1) is nonlinear, the approximate solution can be obtained using the following iterative method: according to Equation (14), the representation of the solution of IDE (1) can be denoted by $u(x) = \sum_{i=1}^{\infty} A_i \bar{\psi}_i(x)$, where

$$A_i = \sum_{k=1}^{i} \beta_{ik} F(x_k, u_{k-1}(x_k), Tu_{k-1}(x_k)).$$

In fact, $A_i$, $i = 1, 2, ..., n.$ are unknown, we will approximate $A_i$ using known $B_i$. For a numerical computations, we define initial function $u_0(x_1)$,
put \( u_0 (x_1) = u (x_1) \), and define the \( n \)-term approximation to \( u (x) \) by:

\[
    u_n (x) = \sum_{i=1}^{\infty} B_i \tilde{\psi}_i (x),
\]

(15)

where the coefficients \( B_i, i = 1, \ldots, n \) are given by

\[
    B_i = \sum_{k=1}^{i} \beta_{ik} F (x_k, u_{k-1} (x_k), T u_{k-1} (x_k)).
\]

(16)

In the iteration process of Equation (15), we can guarantee that the approximation \( u_n (x) \) satisfies the boundary conditions of Equations (1) and (2).

**Corollary 3.1** The approximate solution \( u_n (x) \) is converge uniformly to exact solution \( u (x) \) as \( n \to \infty \).

**Proof.** For any \( x \in [0, 1] \), we have

\[
    |u_n (x) - u (x)| = \left| \langle u_n (x) - u (x), K'_x (x) \rangle \right|_{W^2_2} \\
    \leq \| K'_x (x) \|_{W^2_2} \| u_n (x) - u (x) \|_{W^2_2} \\
    \leq M \| u_n (x) - u (x) \|_{W^2_2}.
\]

where \( M \geq 0 \) is positive constant. Hence, if \( \|u_n (x) - u (x)\|_{W^2_2} \to 0 \) as \( n \to \infty \), then the approximate solution \( u_n (x) \) converge uniformly to exact solution \( u (x) \).

**Theorem 3.2** Assume that \( u (x) \in W^2_2 [0, 1] \) is the solution of boundary value problems (1) and (2) and \( r_n (x) \) is the difference between the approximate solution \( u_n (x) \) and the exact solution \( u (x) \). Then, \( r_n (x) \) is monotone decreasing in the sense of the norm of \( W^2_2 [0, 1] \). i.e. \( r_n \to 0 \) as \( n \to \infty \).

**Proof.** It is obvious that

\[
    ||r_n (x)||_{W^2_2}^2 = ||u (x) - u_n (x)||_{W^2_2}^2 \\
    = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{\infty} \beta_{nk} F (x_k, u_{k-1} (x_k), T u_{k-1} (x_k)) \tilde{\psi}_i (x) \right\|_{W^2_2}^2 \\
    = \left\| \sum_{i=n+1}^{\infty} A_i \tilde{\psi}_i (x) \right\|_{W^2_2}^2 = \sum_{i=n+1}^{\infty} (A_i)^2,
\]

and \( ||r_{n-1} (x)||_{W^2_2}^2 = \sum_{i=n}^{\infty} (A_i)^2 \). Thus, \( ||r_n (x)||_{W^2_2} \leq ||r_{n-1} (x)||_{W^2_2} \). Consequently, the difference \( r_n (x) \) is monotone decreasing in the sense of \( || \cdot ||_{W^2_2} \). So, the proof of the theorem is complete.

The approximate solution \( u_n^N (x) \) can be obtained by taking finitely many terms in the series representation of \( u_n (x) \) and

\[
    u_n^N (x) = \sum_{i=1}^{N} \sum_{k=1}^{i} \beta_{ik} F (x_k, u_{n-1} (x_k), T u_{n-1} (x_k)) \tilde{\psi}_i (x).
\]
4 Numerical Examples

To illustrate the accuracy and applicability of this method for first-order IDE of Volterra type we take some examples in this section. Results obtained are compared with the exact solution of each example and are found to be in good agreement with each other. In the process of computation, all the symbolic and numerical computations performed by using Mathematica 7.0 software package.

**Example 4.1** Consider the linear Volterra integrodifferential equation

\[
\begin{aligned}
&\frac{d}{dx}x(u(x) - t)u(t)dt = 1 + (2x - 1)e^{x^2-x}, \quad 0 \leq x, t \leq 1, \\
&u(0) - u(1) = 0.
\end{aligned}
\]

Using reproducing kernel space method, taking \(x_i = \frac{i-1}{n-1}\), \(i = 1, 2, ..., n\), the relative errors \(\left|\frac{u(x) - u(x)}{u(x)}\right|\) between the approximate solution \(u_n(x)\) and exact solution \(u(x)\) and numerical results at some selected grid points for \(n = 101\) are summarized in Table 1. The exact solution is \(u(x) = e^{x(x-1)}\).

<table>
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<tr>
<th>(x)</th>
<th>Exact sol.</th>
<th>Approximate sol.</th>
<th>Absolute error</th>
<th>Relative error</th>
</tr>
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<td>0.929043</td>
<td>0.92904423211485</td>
<td>9.95299 \times 10^{-7}</td>
<td>1.07131 \times 10^{-6}</td>
</tr>
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<td>0.87424012613232</td>
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<tr>
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<td>0.80444692632266</td>
<td>2.29859 \times 10^{-7}</td>
<td>2.85736 \times 10^{-7}</td>
</tr>
<tr>
<td>0.64</td>
<td>0.794216</td>
<td>0.79420986497894</td>
<td>5.98764 \times 10^{-6}</td>
<td>7.53906 \times 10^{-6}</td>
</tr>
<tr>
<td>0.96</td>
<td>0.962328</td>
<td>0.96232173332145</td>
<td>6.19940 \times 10^{-6}</td>
<td>6.44209 \times 10^{-6}</td>
</tr>
</tbody>
</table>

**Example 4.2** Consider the nonlinear Volterra integrodifferential equation

\[
\begin{aligned}
&\frac{d}{dx}x(2x - 1)(u(x))^2 - \int_0^x \frac{2u(t)}{u(t)}dt = f(x), \quad 0 \leq x, t \leq 1, \\
&u(0) - u(1) = 0.
\end{aligned}
\]

where \(f(x) = x^4(0.5 - 0.4x) - x \left( x + \frac{x-1}{(1-x^2+x^3)} \right) \) such that the exact solution is \(u(x) = \frac{1}{1-x^2+x^4}\). The numerical results at some selected grid points, taking \(x_i = \frac{i-1}{n-1}\), \(i = 1, 2, ..., n\), for \(n = 101\) are summarized in Table 2.

<table>
<thead>
<tr>
<th>(x)</th>
<th>Exact sol.</th>
<th>Approximate sol.</th>
<th>Absolute error</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08</td>
<td>1.005922</td>
<td>1.00592277982561</td>
<td>9.40558 \times 10^{-8}</td>
<td>9.35020 \times 10^{-8}</td>
</tr>
<tr>
<td>0.16</td>
<td>1.021984</td>
<td>1.02197644788242</td>
<td>1.36590 \times 10^{-7}</td>
<td>1.33653 \times 10^{-7}</td>
</tr>
<tr>
<td>0.32</td>
<td>1.07464</td>
<td>1.07484317639168</td>
<td>3.26394 \times 10^{-7}</td>
<td>3.03667 \times 10^{-7}</td>
</tr>
<tr>
<td>0.48</td>
<td>1.13612</td>
<td>1.1361148269051</td>
<td>9.29582 \times 10^{-7}</td>
<td>8.18210 \times 10^{-7}</td>
</tr>
<tr>
<td>0.64</td>
<td>1.17296</td>
<td>1.17295817032545</td>
<td>1.81766 \times 10^{-6}</td>
<td>1.54964 \times 10^{-6}</td>
</tr>
<tr>
<td>0.80</td>
<td>1.14679</td>
<td>1.14678697833333</td>
<td>2.01249 \times 10^{-6}</td>
<td>1.75489 \times 10^{-6}</td>
</tr>
<tr>
<td>0.96</td>
<td>1.03827</td>
<td>1.0382743983075</td>
<td>5.70106 \times 10^{-7}</td>
<td>5.49089 \times 10^{-7}</td>
</tr>
</tbody>
</table>
Example 4.3 Consider the nonlinear Volterra integro-differential equation

\[ \begin{align*}
    u'(x) + \sinh(u(x)) + \int_0^x (x-t) e^{u(t)} dt &= f(x), \quad 0 \leq x, t \leq 1, \\
    u(0) - u(1) &= 0,
\end{align*} \]

where

\[ f(x) = \frac{1}{12} \left( \frac{12(1-x) + x^3(x^3 - 3x^2 + x - 4)}{1 + x - x^2} \right) \]

such that the exact solution is \( u(x) = \ln(x^2 - x + 1) \). Taking \( x_i = \frac{i-1}{n-1}, \; i = 1, 2, \ldots, n \), the numerical results at some selected grid points for \( n = 101 \) are summarized in Table 3.

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact sol.</th>
<th>Approximate sol.</th>
<th>Absolute error</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>-0.0391567</td>
<td>-0.03915637245474</td>
<td>3.42746 \times 10^{-7}</td>
<td>8.75320 \times 10^{-6}</td>
</tr>
<tr>
<td>0.08</td>
<td>-0.0764492</td>
<td>-0.07644847065077</td>
<td>7.01513 \times 10^{-7}</td>
<td>9.17620 \times 10^{-6}</td>
</tr>
<tr>
<td>0.16</td>
<td>-0.1443320</td>
<td>-0.14433092630750</td>
<td>1.44458 \times 10^{-6}</td>
<td>1.00087 \times 10^{-5}</td>
</tr>
<tr>
<td>0.32</td>
<td>-0.2453890</td>
<td>-0.24538635915237</td>
<td>2.80111 \times 10^{-6}</td>
<td>1.14150 \times 10^{-5}</td>
</tr>
<tr>
<td>0.64</td>
<td>-0.2618840</td>
<td>-0.26188132464529</td>
<td>3.05499 \times 10^{-6}</td>
<td>1.16654 \times 10^{-5}</td>
</tr>
<tr>
<td>0.80</td>
<td>-0.1743530</td>
<td>-0.17435157307715</td>
<td>1.81407 \times 10^{-6}</td>
<td>1.04045 \times 10^{-5}</td>
</tr>
<tr>
<td>0.96</td>
<td>-0.0391567</td>
<td>-0.03915637245474</td>
<td>3.42746 \times 10^{-7}</td>
<td>8.75320 \times 10^{-6}</td>
</tr>
</tbody>
</table>

5 Conclusion

In this paper, we construct a reproducing kernel space in which each function satisfies boundary value conditions of considered problems. In this space, a numerical algorithm is presented for solving fourth-order integro-differential equation of Volterra type. The analytical solution is given with series form in \( W_2^2[0,1] \). The approximate solution obtained by present algorithm converges to analytical solution uniformly. The numerical results are displayed to demonstrate the validity of the present algorithm.

References


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