On Continuity of Algebraic Operations in the Gelfand Topologies Generated by Algebras of Analytic Functions on Banach Spaces

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Abstract
The paper is devoted to study of continuity of the operation of sum (resp. multiplication) on Banach spaces (resp. algebras) in the Gelfand topology generated by some algebras of analytic functions.

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1 Introduction
Let $X$ be a real Banach space and $X_C$ be its complexification (see e.g. [3]). A polynomial $P : X \to \mathbb{R}$ is said to be separating if

$$\inf_{||x||=1} |P(x) - P(0)| > 0.$$ 

An entire (analytic) function $f : X \to \mathbb{R}$ is $r$-uniformly analytic, $r > 0$, if for every $x \in X$ it is bounded at the open ball $B_r(x)$ centered at $x$ of radius $r$. Finally, an entire function $f : Y \to \mathbb{R}$ is called separating if for some $\alpha > 0$,

$$\emptyset \notin \{x \in X : f(x) < \alpha\} \subset B_1(0).$$
Separating polynomials were introduced in [5] and investigated in [6]. Separating analytic functions were proposed by Boiso and Hajek in [3]. For a given complex Banach spaces $Z$ we denote by $H_b(Z)$ the Fréchet algebra of all entire functions of bounded type (i.e. bounded on bounded subsets) and by $H_u^r(Z)$ the algebra of all $r$-uniformly analytic functions on $Z$. Clearly $H_b(Z) \subset H_u^r(Z)$ and $H_u^r(Z)$ is a topological algebra. By $H_b$-topology ($H_u^r$-topology respectively) we mean the Gelfand topology of $H_b(Z)$ ($H_u^r(Z)$ respectively) restricted to $Z$.

Algebras $H_b(Z)$ and their spectra for various spaces $Z$ were investigated by many authors (see e.g. [1], [2], [9], [7]). In particular in [1] it is shown that the operation of sum in $\ell_2$ is discontinuous in $H_b(\ell_2)$-topology. In this paper we investigate conditions of continuity of sum and multiplication in $H_b(Z)$- and $H_u^r(Z)$-topologies. Also we consider the continuity of multiplications in $\ell_{2n}$ and $c_0$ in the same topologies. Note that some related results in the case when $Z$ is a Banach algebra was obtained in [8].

2 Main Results

Theorem 2.1 If $X$ admits a separating polynomial $P$ and $\dim X = \infty$, then the operation of sum is discontinuous on $X_C$ in $H_b$-topology.

Proof. Let $(x_{\alpha})$ be a net in $X_C$ such that $\|x_{\alpha}\| = 1$ and $(x_{\alpha})$ is $H_b$-convergent to 0. Such a net must exist according to [4]. Let $P_C$ be the complexification of $P$ and $\overline{x_{\alpha}}$ is the complex conjugated to $x_{\alpha}$. Then $\overline{x_{\alpha}}$ is $H_b$ convergent to 0 too and $x_{\alpha} + \overline{x_{\alpha}} \in X$ and $\frac{1}{i}(x_{\alpha} - \overline{x_{\alpha}}) \in X$. Let

$$\text{Re}(x_{\alpha}) = \frac{x_{\alpha} + \overline{x_{\alpha}}}{2} \quad \text{and} \quad \text{Im}(x_{\alpha}) = \frac{x_{\alpha} - \overline{x_{\alpha}}}{2i}.$$ 

If $\|\text{Re}(x_{\alpha_{\beta}})\| \rightarrow 0$ on a subnet $(x_{\alpha_{\beta}})$, then

$$P_C(x_{\alpha_{\beta}}) \rightarrow P_C(\text{Im}(x_{\alpha_{\beta}})) = P(\text{Im}(x_{\alpha_{\beta}})).$$

If $P(\text{Im}(x_{\alpha_{\beta}})) \rightarrow 0$ then $\|\text{Im}(x_{\alpha_{\beta}})\| \rightarrow 0$ since $P$ is separating. So $\|x_{\alpha_{\beta}}\| \rightarrow 0$ that contradicts our assumption. So there is a subnet $(x_{\alpha_{\beta}})$ such that $\inf_{x_{\alpha_{\beta}}} \|\text{Re}(x_{\alpha_{\beta}})\| > 0$ or $\inf_{x_{\alpha_{\beta}}} \|\text{Im}(x_{\alpha_{\beta}})\| > 0$. Let us suppose that $\inf_{x_{\alpha_{\beta}}} \|\text{Re}(x_{\alpha_{\beta}})\| > 0$. Then

$$P\left(\frac{x_{\alpha_{\beta}} + \overline{x_{\alpha_{\beta}}}}{2}\right) = P(\text{Re}(x_{\alpha_{\beta}})) \not\rightarrow 0$$

while $(x_{\alpha_{\beta}})$ and $(\overline{x_{\alpha_{\beta}}})$ are both $H_b$-convergent to 0. So the operation of sum is discontinuous. The same work is in the case if $\inf_{x_{\alpha_{\beta}}} \|\text{Im}(x_{\alpha_{\beta}})\| > 0$. 

On continuity of algebraic operations

Note that real $\ell_{2n}$ and $L_{2n}$ admit separating polynomials for all positive integer $n$, namely $P(x) = ||x||^{2n}$, while $c_0$ does not admit separating polynomials (see e.g. [3]).

In the case when $X = c_0$ we know that $H_b$-topology coincides with the weak topology on bounded sets on $c_0$. So the sum is $H_b$-topology continuous on $c_0$.

Let us consider the following analytic function

$$d(x) = \sum_{n=1}^{\infty} x_n^{2(2n-1)}$$

which belongs to $H_u^1(c_0)$ (cf. [3]). Denote by $A_0$ the minimal Fréchet algebra which contains $H_b(c_0)$ and $d(x)$. Clearly $H_b(c_0) \subset A_0 \subset H_u^1(c_0)$. Let $x_n = e_{2n-1} + ie_{2n}$, where $\{e_n\}$ is the standard basis in $c_0$. Then $\{x_n\}$ weakly converges to 0 and $d(x_n) = 0$ so $\{x_n\}$ converges to zero in the Gelfand topology of $A_0$. On the other hand $d(x_n + \overline{x}_n) = d(2e_{2n-1}) = 2^{4(2n-1)-2} \not\to 0$. Hence, we have proven the following theorem.

**Theorem 2.2** The operation of sum is discontinuous on $c_0$ in $A_0$-topology.

**Open question**: Does the operation of sum continuous on $c_0$ in $H_u^1$-topology?

Let us consider $\ell_{2n}$ as a Banach algebra with the pointwise multiplication:

$$\sum_{k=1}^{\infty} x_k e_k \sum_{k=1}^{\infty} y_k e_k = \sum_{k=1}^{\infty} x_k y_k e_k,$$

where $\{e_k\}$ is the standard basis in $\ell_{2n}$, $n \in \mathbb{N}$.

**Theorem 2.3** The operation of multiplication in complex Banach algebra $\ell_{2nc}$ is discontinuous in $H_b$-topology.

**Proof.** Let $(x_\alpha)$ be a net in $\ell_{2nc}$ such that $||x_\alpha|| = 1$ and $x_\alpha \to 0$ in $H_b$-topology. Let $P_\mathbb{C}$ be a polynomial on $\ell_{2nc}$ of the form

$$P_\mathbb{C}(x) = \sum_{k=1}^{\infty} x_k^n.$$ 

Then

$$P_\mathbb{C}(x_\alpha \overline{x}_\alpha) = \sum_{k=1}^{\infty} (x_{\alpha k} \overline{x}_{\alpha k})^n = \sum_{k=1}^{\infty} |x_{\alpha k}|^{2n} = ||x_\alpha||^{2n} = 1 \not\to 0.$$ 

By the same way we can prove a similar result on algebra $c_0$. Let function $d(x)$, sequence $\{x_n\}$ and algebra $A_0$ be as above. Then

$$d(x_n \overline{x}_n) = d(e_{2n-1}) + d(e_{2n}) = 2 \not\to 0.$$ 

So we proven the following theorem.

**Theorem 2.4** The operation of product is discontinuous on $c_0$ in $A_0$-topology.
References


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