Approximation to the Dissipative Klein-Gordon Equation

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Abstract

In this paper we show that there exists a solution of the cubic nonlinear Klein-Gordon equation for a small parameter. We construct a traveling wave equation and we show that the corresponding system does not have periodic orbits for some real constants.

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1 Introduction

It was made some applications to the nonlinear Klein-Gordon equation [1]. It was found an antibound state for the Klein-Gordon equation [2]. In [3] was used a method with Green functions for constructing asymptotics of eigenvalues for the linear Klein-Gordon equation. In [4] was studied the Klein-Gordon equation. In [5] was given an explicit formula for the eigenvalue below the essential spectrum of discrete Klein-Gordon operator. We show there exists a solution for a small parameter and construct a traveling wave equation and a system without periodic orbits.

2 Preliminary Notes

The Klein-Gordon equation

\[ u_{tt} - \Delta u + \alpha u_t + \beta u + \gamma u^3 = 0 \] (1)

where \( \alpha, \beta, \gamma \) are real constants.

3 Main Results

These are the main results of the paper.

**Theorem 3.1.** The solution of (1) is (5) for a small parameter \( \gamma \).

**Proof.** Taking Fourier transform we have

\[ \hat{u}_{tt}(p,t) + (p^2 + \beta) \hat{u}(p,t) + \alpha \hat{u}_t + \gamma \hat{u}^2 * \hat{u} = 0 \] (2)

with solution

\[
\hat{u}(p,t) = e^{\frac{1}{2}t\sqrt{\alpha^2 - 4\beta - 4p^2}} \int_0^t \frac{\gamma f(p,\zeta) e^{\frac{1}{2}t(\sqrt{\alpha^2 - 4\beta - 4p^2} + \alpha \zeta)}}{\sqrt{\alpha^2 - 4\beta - 4p^2}} d\zeta + \\
+ k_1(p) e^{\frac{1}{2}t(-\sqrt{\alpha^2 - 4\beta - 4p^2} - \alpha)} + k_2(p) e^{\frac{1}{2}t(\sqrt{\alpha^2 - 4\beta - 4p^2})} (3)
\]

where \( f = \hat{u}^2 * \hat{u} \). Rewriting this equation we have

\[ \hat{u}(p,t) = \gamma T \hat{u}(p,t) + k_1(p) e^{\frac{1}{2}t(-\sqrt{\alpha^2 - 4\beta - 4p^2} - \alpha)} + k_2(p) e^{\frac{1}{2}t(\sqrt{\alpha^2 - 4\beta - 4p^2})} (4)\]
where
\[
\hat{u}(p, t) = e^{\frac{1}{2}t(\sqrt{\alpha^2 - 4\beta - 4p^2} - \alpha)} \int_0^t - \frac{f(p, \zeta)e^{\frac{1}{2}\zeta(\sqrt{\alpha^2 - 4\beta - 4p^2} - \alpha)}}{\sqrt{\alpha^2 - 4\beta - 4p^2}} d\zeta +
\]
\[
e^{\frac{1}{2}t(-\sqrt{\alpha^2 - 4\beta - 4p^2} - \alpha)} \int_0^t f(p, \zeta)e^{\frac{1}{2}\zeta(\sqrt{\alpha^2 - 4\beta - 4p^2} + \alpha)} \sqrt{\alpha^2 - 4\beta - 4p^2} d\zeta +
\]
\[
e^{\frac{1}{2}t(\sqrt{\alpha^2 - 4\beta - 4p^2} - \alpha)} \int_0^t - \frac{f(p, \zeta)e^{\frac{1}{2}\zeta(\sqrt{\alpha^2 - 4\beta - 4p^2} + \alpha)}}{\sqrt{\alpha^2 - 4\beta - 4p^2}} d\zeta +
\]
\[
e^{\frac{1}{2}t(-\sqrt{\alpha^2 - 4\beta - 4p^2} + \alpha)} \int_0^t f(p, \zeta)e^{\frac{1}{2}\zeta(\sqrt{\alpha^2 - 4\beta - 4p^2} - \alpha)} \sqrt{\alpha^2 - 4\beta - 4p^2} d\zeta.
\]

Then by Neumann series
\[
\hat{u}(p, t) = \sum_{n=0}^{\infty} \gamma^n T^n(k_1(p)e^{\frac{1}{2}t(\sqrt{\alpha^2 - 4\beta - 4p^2} - \alpha)} + k_2(p)e^{\frac{1}{2}t(\sqrt{\alpha^2 - 4\beta - 4p^2} + \alpha)})
\]
(5)

\[
\square
\]

4 Traveling wave solution

If the Klein-Gordon equation has a traveling wave solution of the form

\[
u(x, y, t) = u(\xi), \quad \xi = kx + ly - \lambda t
\]

where \(k, l, \lambda\) are real constants. Substituting (6) into (1) we obtain

\[
u_{\xi\xi} - \frac{\alpha\lambda}{\lambda^2 - k^2 - l^2} u_{\xi} + \frac{\beta}{\lambda^2 - k^2 - l^2} u + \frac{\gamma}{\lambda^2 - k^2 - l^2} u^3 = 0
\]

(7)

5 Dynamical system

Taking \(u_\xi = y\) and \(u = x\) we have the following system

\[
\begin{cases}
  x_\xi = y \\
  y_\xi = \sigma y - \mu x - \nu x^3
\end{cases}
\]

where \(\sigma = \frac{\alpha\lambda}{\lambda^2 - k^2 - l^2} < 0, \quad \mu = \frac{\beta}{\lambda^2 - k^2 - l^2} > 0, \quad \nu = \frac{\gamma}{\lambda^2 - k^2 - l^2} > 0\). The Hamiltonian system

\[
\begin{cases}
  x_\xi = -H_y \\
  y_\xi = H_x
\end{cases}
\]

(9)

has this solution \(H = -y^2 + \sigma xy - \mu \frac{x^2}{2} - \nu \frac{x^4}{4} = K\) for some constant \(K\).

For the next results we are going to use the Poincaré-Bendixson theorem and the following quasi-differential equation

\[
f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} = h \left[ C(x_1, x_2) - \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \right].
\]

(10)
Theorem 5.1. The dynamical system (8) can be generalized to (11) and both do not have periodic orbits for $y + \sigma > 0$ and $x \in \mathbb{R}$.

Proof. Taking $K = 0$ and $x_\xi = y$, and supposing that $\frac{\partial h}{\partial x_2} = 0$, $C(x_1, x_2) = \sigma + y > 0$. Using (10) we have $y \frac{\partial h}{\partial x_1} = h [C(x_1, x_2) - \sigma]$ and $\frac{\partial h}{\partial x_1} = h$. So $h = e^x$. If $\frac{\partial f_2}{\partial x_2} = \sigma$ then $f_2 = \sigma y + C_2(x)$. From equation (10), we get the ordinary differential equation $f_1 = \sigma + y - (\frac{\partial f_1}{\partial x_1} + \sigma)$. Then its solution is $f_1 = y + c_1(y)e^{-x}$. We obtain the generalized dynamical system

$$\begin{aligned}
&x_1 = y + c_1(y)e^{-x} \\
x_2 = \sigma y + C_2(x).
\end{aligned}$$

(11)

Taking $\nu = 1$ into (8) and using the following Poincaré transformation

$$\frac{dt}{z^2} = d\tau, \quad x_1 = \frac{1}{z}, \quad x_2 = \frac{u}{z}, \quad (z \neq 0)$$

we obtain

$$\begin{aligned}
u \tau &= -u^2z^2 - \mu z^2 - 1 + \sigma uz^2 \\
\tau &= -uz^3.
\end{aligned}$$

(12)

Theorem 5.2. The system (13) can be generalized to (14) and both do not have periodic orbits for $x_2 > 0$.

Proof. Taking $x_1 = u$, $x_2 = z$. Suppose $\frac{\partial f_1}{\partial x_1} = -2x_1x_2^2 + \sigma x_2^2$, then $f_1 = -x_1^2x_2^2 + \sigma x_2^3x_1 + C_1(x_2)$. From (10), and taking $C = \sigma x_2^3 < 0$ and $h = \frac{1}{x_2^3}$ with $x_2 > 0$. Then $\frac{\partial h}{\partial x_2} = -\frac{5}{x_2^4}$ and we have an ordinary differential equation

$$2x_1x_2^3 - x_2 \frac{\partial f_2}{\partial x_2} = -5f_2.$$

Then its solution is

$$f_2 = C_2(x_1)x_2^5 - x_1x_2^3$$

Also, it holds

$$\frac{\partial}{\partial x_1}(f_1h) + \frac{\partial}{\partial x_2}(f_2h) = \frac{\sigma}{x_2^3} > 0.$$

We have the following generalized dynamical system

$$\begin{aligned}
x_1 &= -x_1^2x_2^2 + \sigma x_2^3x_1 + C_1(x_2) \\
x_2 &= C_2(x_1)x_2^5 - x_1x_2^3.
\end{aligned}$$

(14)

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References


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