The Solution of Radial Diffusivity and
Shock Wave Equations by
Elzaki Variational Iteration Method

Tarig M. Elzaki
King Abdulaziz University
Department of Mathematics
Jeddah 21589, Saudi Arabia

Hwajoon Kim∗
Kyungdong University
School of IT Engineering
Yangju 482-010, Gyeonggi, Korea
∗Corresponding author

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Abstract
In this work, we present a reliable combined Elzaki transform and
the new modified variational iteration method to solve radial diffusivity
and shock wave equations. The analytical results of these equations
have been obtained in terms of convergent series with easily computable
components. The nonlinear terms in these equations can be handled by
using the new modified variational iteration method.

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sivity equation, shock wave equation
1 Introduction

Nonlinear equations are of the most of important phenomena across the world. Nonlinear phenomena have important efficiency on applied mathematics, physics, and issues related to engineering. The importance of obtaining the exact solution of nonlinear radial diffusivity and shock wave equations in physics and applied mathematics is still a big problem that needs new methods to discover exact or approximate solutions.

The concept of Elzaki transform[6-13] was proposed by Tarig M. Elzaki in 2011, and in the recent years, many authors mainly had paid attention to studying the solution of nonlinear partial differential equations by using various methods. Those are the adomian decomposition method[15], homotopy perturbation method [7, 17, 20-23], variational iteration method, differential transform method [2, 4, 6, 14, 23], and projected differential transform method [8, 16], and so on.

In this work, we will use the new modified form of the variational iteration method, Elzaki variational iteration method, and this method provides an effective and efficient way of solving a wide range of nonlinear operator equations. The advantage of this method is its capability of combining two powerful methods for obtaining exact or approximate solutions in nonlinear radial diffusivity and shock wave equations. This article has considered the effectiveness of the Elzaki variational iteration method in solving nonlinear radial diffusivity and shock wave equations.

Consequently, the proposed method is applied in a direct way without using linearization. The fact that the proposed method is successfully implemented by using the initial conditions only, and solving the nonlinear radial diffusivity and shock wave equations without using Adomian’s polynomials.

2 The Elzaki transform and variational iteration method

The basic definition of Elzaki transform is defined as

$$E[f(t)] = v \int_{0}^{\infty} f(t)e^{-t/v} dt$$

for $t > 0$. Tarig M. Elzaki and Sailh M. Elzaki in [11-14] proposed the modified Sumudu transform[4-5, 18-19, 24], namely Elzaki transform, and were applied to partial differential equations, ordinary differential equations, system of ordinary and partial differential equations and integral equations[1]. Elzaki transform has a strong point for solving some differential equations which cannot be solved by Sumudu transform[11]. We can obtain the following formulas
from the definition of Elzaki transform. To obtain Elzaki transform of partial derivative we use integration by parts, and then we have

\[ a) \quad E\left[ \frac{\partial f(x,t)}{\partial t} \right] = \frac{1}{v} T(x,v) - v f(x,0) \]

\[ b) \quad E\left[ \frac{\partial^2 f(x,t)}{\partial t^2} \right] = \frac{1}{v^2} T(x,v) - f(x,0) - v \frac{\partial f(x,0)}{\partial t} \]

\[ c) \quad E\left[ \frac{\partial f(x,t)}{\partial x} \right] = \frac{d}{dx} [T(x,v)] \]

\[ d) \quad E\left[ \frac{\partial^2 f(x,t)}{\partial x^2} \right] = \frac{d^2}{dx^2} [T(x,v)] \]

for \( E(f(t)) = T(v) \). We can easily extend this result to the \( n \)-th partial derivative by using the induction. The details with respect to Elzaki transform can be found in [7]. On the other hand, to illustrate the basic concept of the variational iteration method[7], let us consider the following general differential equation

\[ L[u(x,t)] + N[u(x,t)] = g(x,t), \quad u(x,0) = h(x), \tag{2} \]

where \( L \) is a linear operator of the first order, \( N \) is a nonlinear operator and \( g(x,t) \) is inhomogeneous term. According to the variational iteration method, we can construct a correction functional as

\[ u_{n+1} = u_n + \int_0^t \lambda [Lu_n(x,s) + N\tilde{u}_n(x,s) - g(x,s)] \, ds, \]

where \( \lambda \) is a Lagrange multiplier (\( \lambda = -1 \)), the subscript \( n \) denotes the \( n \)-th approximation, \( \tilde{u}_n \) is considered as a restricted variation, i.e. \( \delta \tilde{u}_n = 0 \). The successive approximation \( u_{n+1} \) of the solution \( u \) will be readily obtained upon using the determined Lagrange multiplier and any selective function \( u_0 \). Consequently, the solution is given by \( u = \lim_{n \to \infty} u_n \).

### 3 The Elzaki variational iteration method

In this work, we assume that \( L \) is an operator of the first order \( \partial/\partial t \) in the equation (2). Let us take Elzaki transform of both sides and apply the differentiation property of Elzaki transform. Then we have

\[ E[Lu(x,t)] + E[Nu(x,t)] = E[g(x,t)] \]

and

\[ E[u(x,t)] = vE[g(x,t)] + vh(x) - vE[Nu(x,t)]. \]

Applying the inverse Elzaki transform of both sides of the equation, we have

\[ u(x,t) = G(x,t) - E^{-1}\{vE[Nu(x,t)]\}, \]
where \( G(x,t) \) represents the terms arising from the source term and the prescribed initial condition. Taking the first partial derivative with respect to \( t \), we have

\[
  u_t(x,t) - \frac{\partial}{\partial t}G(x,t) + \frac{\partial}{\partial t}E^{-1}\{vE[Nu(x,t)]\} = 0.
\]

By the correction function of the irrational method, we have

\[
  u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{ (u_n)_s(x,s) - \frac{\partial}{\partial s}G(x,s) + \frac{\partial}{\partial s}E^{-1}\{vE[Nu_n(x,s)]\} \} ds,
\]

or alternately

\[
  u_{n+1}(x,t) = G(x,t) - E^{-1}\{vE[Nu_n(x,t)]\}.
\]

Thus, we can obtain the solution \( u \) by \( u(x,t) = \lim_{n \to \infty} u_n(x,t) \).

## 4 Illustrative problems

Would let us solve some examples by using the Elzaki variational iteration method.

**Example 4.1** Consider the nonlinear radial diffusivity equation

\[
  \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r}
\]

with \( f(r,0) = r \).

**Solution.** Taking Elzaki transform on the given equation, we have

\[
  E[f(r,t)] = rv^2 + vE\left[\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r}\right]
\]

because of (a) of page 2. Since \( E(1) = v^2 \), the inverse Elzaki transform implies that

\[
  f(r,t) = r + E^{-1}\{vE\left[\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r}\right]\}.
\]

According to the equation (3), the correction function is given by

\[
  f_{n+1}(r,t) = f_n(r,t) + E^{-1}\{vE\left[\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r}\right]\}, \quad f_0(r,t) = r.
\]

Now let us apply the the Elzaki variational iteration method, the solution in series is given by

\[
  f_0(r,t) = r
\]
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\[ f_1(r,t) = r + E^{-1}\{vE\left(\frac{1}{r}\right)\} = r + E^{-1}\left(\frac{1}{r}\right) = r + \frac{1}{r}t \]
\[ f_2(r,t) = r + \frac{1}{r}t + E^{-1}\{vE\left(\frac{2}{r^3}t - \frac{1}{r^3}t\right)\} = r + \frac{1}{r}t + \frac{1}{2r^3}t^2 \]

because of \( E(t^n) = n!u^{n+2} \). Continue this process, and we can obtain the solution

\[ f(t,r) = r + \frac{t}{r} + \frac{t^2}{2r^3} + \frac{3t^3}{2r^5} + \frac{75t^4}{8r^7} + \cdots = r + \sum_{x=1}^{\infty} \frac{1^2 \times 3^2 \times \cdots (2x-1)^2}{x!r^{2x-1}}t^x. \]

**Example 4.2** Consider the nonlinear shock wave equation

\[ u_t(x,t) + \left(\frac{5}{2} + \frac{5}{16}u\right)u_x = 0 \]

with \((x,t) \in R \times [0,T] \), \( u(x,0) = e^{-\frac{x^2}{2}} \).

**Solution.** In a similar fashion with example 4.1, let us take Elzaki transform on both sides. Then we have

\[ E[u(x,t)] = v^2e^{-\frac{x^2}{2}} - vE\{\frac{1}{2} + \frac{5}{16}u\}. \]

Similarly, let us take the inverse Elzaki transform to find that

\[ u(x,t) = e^{-\frac{x^2}{2}} - E^{-1}\{vE\{u_x(\frac{1}{2} + \frac{5}{16}u)\}\}. \]

By the equation (3), the correct function is given by

\[ u_{n+1}(x,t) = u_n(x,t) - E^{-1}\{vE\{u_{nx}(\frac{1}{2} + \frac{5}{16}u)\}\}. \]

Hence, the solution in series form is given by

\[ u_0(x,t) = e^{-\frac{x^2}{2}} \]
\[ u_1(x,t) = e^{-\frac{x^2}{2}} - E^{-1}\{vE[-\frac{1}{2}xe^{-\frac{x^2}{2}} - \frac{5}{16}xe^{-x^2}]\} = e^{-\frac{x^2}{2}}[1 + \frac{1}{2} xt + \frac{5}{16} xte^{-\frac{x^2}{2}}]. \]

Thus, continue this process, and we obtain the series solution

\[ u(x,t) = e^{-\frac{x^2}{2}}[1 + \frac{1}{2} xt + \frac{5}{16} xte^{-\frac{x^2}{2}} + \cdots]. \]
References


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