Number of Limit Cycles for Homogeneous Polynomial System

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Abstract

In this paper the bifurcation of limit cycles at infinity for a class of homogeneous polynomial system of degree four is examined. This requires a problem for bifurcation of limit cycles at infinity be converted from the original system to the class of complex autonomous differential system. The evaluation of the conditions from the origin to be a centre and the highest degree fine focus results from the calculation of singular point values. A quartic system is constructed for which it can bifurcate with only one limit cycle at infinity when the normal parameters are constant.

Keywords: Infinity; Singular point quantities; Quartic differential system; Centre condition; Bifurcation of limit cycles

1. Introduction

In this work, the bifurcation of limit cycles is being referred to particularly to the bifurcation at infinity. The computation of the focal values is one way to examine it. In the case of bifurcation of limit cycles at the origin, a lot of work has been done, most of them have been reported in [6, 11]. For the case at infinity, the study is concerned with the following system of degree $2n + 1$, [6]
where \( X_k(x, y) \), \( Y_k(x, y) \) are homogeneous polynomials of degree \( k \) of \( x, y \). For this system, the equator \( \Gamma_\infty \) on the Poincare closed sphere is a trajectory of the system, having no real singular point, and \( \Gamma_\infty \) is called infinity of the system. Concerning the number of limit cycles bifurcated from infinity (large limit cycles), there are some related results in the literature as follows: cubic system, with four limit cycles in [7], five limit cycles in [1], six limit cycles in [8]; quantic system, with five limit cycles in [10], six limit cycles in [2], seven limit cycles in [12], eight limit cycles in [5] and nine limit cycles in [4].

In this work, the following real polynomial differential system is considered:

\[
\begin{align*}
\frac{dx}{dt} &= X_1(x, y) + X_4(x, y), \\
\frac{dy}{dt} &= Y_1(x, y) + Y_4(x, y),
\end{align*}
\]

(2)

where \( X_k = \sum_{i=0}^{k} a_i x^{k-i} y^i \), \( Y_k = \sum_{i=0}^{k} b_i x^{k-i} y^i \), are \( k \)-th homogeneous polynomials, \( k = 1, 4; a_i \) and \( b_j \in \mathbb{R} \). In Section 2, some characteristics of system (2) and certain known results are outlined. In Section 3, a recursive method for calculating singular point quantities at infinity is deduced and computed for the first eight singular point quantities at infinity of system (6). The recursive method is linear and it avoids complex integrating operations, and can be readily done by using a computer algebra system such as Mathematica (version 13). Using the recursive method, the first eight singular point quantities at infinity of (6) are obtained. The conditions at infinity to be a centre are given as well. In Section 4, a new system from the original system (2) is constructed.

2. Polynomial Differential System and Some Preliminary Results

The system (2) transforms to polar coordinates by using \( x = \cos \theta, y = \sin \theta \), and after multiplying system (2) by \( \cos \theta \) and \( \sin \theta \) then adding and subtracting the resulting equations we obtain respectively
Number of limit cycles for homogeneous polynomial system

\[
\dot{r} = \cos \theta (X_1(r, \theta) + X_4(r, \theta)) + \sin \theta (Y_1(r, \theta) + Y_4(r, \theta))
\]
\[
\dot{\theta} = \frac{1}{r} \left( \cos \theta (Y_1(r, \theta) + Y_4(r, \theta)) - \sin \theta (X_1(r, \theta) + X_4(r, \theta)) \right). \tag{3}
\]

Taking the form

\[
\frac{dr}{d\theta} = r \left( \frac{\cos \theta (X_1(r, \theta) + X_4(r, \theta)) + \sin \theta (Y_1(r, \theta) + Y_4(r, \theta))}{\cos \theta (Y_1(r, \theta) + Y_4(r, \theta)) - \sin \theta (X_1(r, \theta) + X_4(r, \theta))} \right), \tag{4}
\]

and writing the solution of (4) with \( R(r, 0) = h \) as the initial value, then

\[
R(r, \theta) = \sum_{m=1}^{\infty} v_m(\theta) h^m,
\]

where \( v_m(0) = 0, m=1,2,... \).

The system (2) can now be transformed into a class of complex autonomous differential system, by means of the transformation

\[
z = x + yi, \quad w = x - yi, \quad T = it \quad i = \sqrt{-1}, \tag{5}
\]

thus obtaining

\[
\frac{dz}{dT} = A_{1,0} z + A_{0,1} w + A_{4,0} z^4 + A_{3,1} z^3 w + A_{2,2} z^2 w^2 + A_{1,3} z w^3 + A_{0,4} w^4,
\]
\[
\frac{dw}{dT} = B_{1,0} z + B_{0,1} w + B_{4,0} z^4 + B_{3,1} z^3 w + B_{2,2} z^2 w^2 + B_{1,3} z w^3 + B_{0,4} w^4, \tag{6}
\]

where

\[
A_{1,0} = \frac{1-i}{2}, \quad A_{0,1} = \frac{1+i}{2}, \quad B_{1,0} = \frac{-1-i}{2}, \quad B_{0,1} = \frac{-1+i}{2},
\]
\[
A_{4,0} = \frac{a_1 - a_2 i + a_4 i + a_5 + b_1 + b_2 i + b_3 + b_4 i + b_5}{8},
\]
\[
B_{4,0} = \frac{a_1 - a_2 i + a_4 i + a_5 + b_1 - b_2 i - b_3 - b_4 i - b_5}{8}, \tag{7}
\]
\[
A_{3,1} = \frac{4a_1 - 2a_2 i - 2a_4 i - 4a_5 - 4b_1 + 2b_2 i + 2b_4 i + 4b_5}{8},
\]
According to Theorem 3.2* in [6], the following lemma is used for further evaluation.

**Lemma 2.1:** For the system (6) it can be successively developed in terms of a formal series

\[ F(z, w) = \frac{1}{zw} \sum_{k=0}^{\infty} \frac{f_k(z, w)}{(zw)^k}, \]

such that

\[ \frac{dF}{dT} = \sum_{m=0}^{\infty} \frac{\mu(m)}{(zw)^{m-2}}, \]

where \( f_k(z, w) = \sum_{\alpha+\beta=k} C_{\alpha,\beta} z^\alpha w^\beta \) is a homogeneous polynomial of degree \( k \) with \( C_{0,0} = 1, C_{k,k} = 0, k = 1, 2, \ldots \).

**Proof:** The proof is given in [12].

From the above lemma, \( v_{2m+1}(2\pi) \), is the \( m-th \) focal value at infinity of system (2), and \( v_{2m+1}(2\pi) \approx i\pi\mu(m) \), where \( \mu(m) \) is \( m-th \) singular point quantity at infinity.
of system (6), and "\( \approx \)" is the algebraic equivalence. Now \( v_{2m+1}(2\pi) \approx i\pi\mu(m) \) represents the existing \( \xi^{(k)} \), i.e.

\[
v_{2m+1}(2\pi) = i\pi(\mu(m) + \sum_{k=m}^{m+1} \xi^{(k)} \mu(k)) .
\]

(10)

The focal values of system (2) can be deduced from the singular point quantities of system (6). It is obvious that the infinity is the centre of system (2) if and only if all \( \mu(m) = 0, m = 1, 2, \ldots \).

3. The Computation of Singular Point Quantities and Centre Condition at Infinity

From lemma 2.1, a recursive formula can be deduced to calculate singular point quantities of the infinity of system (6).

**Theorem 3.1:** In the system (6), for any integer \( m, \mu(m) \) is determined by the following formula

\[
C_{a,\beta} = \frac{1}{\beta - \alpha} \sum_{k+j=4}^{a+\beta+3} [(\alpha + 1)A_{k,j-1} - (\beta + 1)B_{j,k-1}]C_{a-k+1,\beta+j+1}
\]

\[
\mu(m) = \sum_{k+j=4}^{2m+2} (A_{k,j} - B_{j,k})C_{m-k+1,m-j+1} ,
\]

(11)

with the following conditions

\( C_{0,0} = 1 \),

where \( (\alpha = \beta > 0) \) or \( \alpha < 0, \beta < 0, C_{a,\beta} = 0 \).

(12)

**Proof:** According to lemma 2.1, differentiating both sides of (8) with respect to \( T \) along the trajectories of system (6), would result in

\[
\frac{dF}{dT} = \sum_{n=1}^{\infty} (z^j)^{2n-\mu} \left[ \frac{\partial f_m}{\partial z} z - \frac{\partial f_m}{\partial w} w + \sum_{\alpha,\beta,\gamma} d_{\alpha,\beta,\gamma} z^\alpha w^\beta \right]
\]

(13)
where

\[
\sum_{\alpha \neq \beta \neq \mu} d_{\alpha, \beta} z^{\alpha} w^{\beta} = \left[ (\frac{\partial f_{m-2}}{\partial z} z + (1 - m) f_{m-2}) \right] \sum_{i=0}^{3} a_i - (\frac{\partial f_{m-2}}{\partial w} w + (1 - m) f_{m-2}) \sum_{i=0}^{3} b_i k^{i+1} w^{i+1}
\]

\[
+ \left[ (\frac{\partial f_{m-4}}{\partial z} z + (3 - m) f_{m-4}) \right] \sum_{i=0}^{2} a_i - (\frac{\partial f_{m-4}}{\partial w} w + (3 - m) f_{m-4}) \sum_{i=0}^{2} b_i k^{i+2} w^{i+2}
\]

\[
+ \left[ (\frac{\partial f_{m-6}}{\partial z} z + (5 - m) f_{m-6}) \right] \sum_{i=0}^{1} a_i - (\frac{\partial f_{m-6}}{\partial w} w + (5 - m) f_{m-6}) \sum_{i=0}^{1} b_i k^{i+3} w^{i+3}.
\]

From (9) and (13), for any \( \alpha, \beta \), where \( \alpha \neq \beta \), letting \( C_{\alpha, \beta} = \frac{d_{\alpha, \beta}}{\beta - \alpha} \) and \( \mu(m) = d_{m,m} \), then the proof is complete.

Applying the recursion formulae (11) and (12), the singular point quantities are computed and simplified at infinity of system (6), and then the following result is generated.

The first eight singular point quantities of infinitesimal quantities of system (6) are given as follows (refer to the Appendix)

\[
\mu(1) = 0, \\
\mu(2) = 0, \\
\mu(3) = B_{1,3}B_{4,0} - A_{3,1}A_{4,0} - A_{2,2}A_{4,1} - 2A_{2,2}^2 + 5A_{2,2}B_{3,1} - 2B_{3,1}^2, \\
\mu(4) = 0, \\
\mu(5) = 0,
\]

\[
\mu(6) = \frac{19}{15} A_{2,2}^3 A_{3,1} B_{4,0} - \frac{1}{3} A_{2,2}^2 B_{3,2} B_{4,0} + \frac{32}{3} A_{2,2}^2 A_{4,0} B_{3,1} - 30A_{2,2}^2 A_{3,1} + \frac{1}{2} A_{2,2}^2 B_{3,2} B_{3,1} + 24A_{2,2}^2 A_{3,1} B_{2,2} - 12A_{2,2}^2 B_{2,2}^2 - 6A_{2,2}^2
\]

\[
B_{3,1} + 30A_{2,2}^2 A_{4,0} B_{3,1} + \frac{35}{2} B_{3,1}^3 B_{4,0} + 20A_{2,2}^2 B_{3,2} B_{4,0} + 6B_{2,2}^2 B_{2,3} + 45B_{2,2}^2 B_{3,1} + 67A_{2,2}^2 B_{2,2} + \frac{12}{5} A_{2,2} B_{2,2} B_{4,0} + 8A_{2,2} A_{3,1} B_{3,1} - 25B_{2,2}^2
\]

\[
- \frac{69}{10} B_{3,1} B_{2,2}^2 + 10A_{2,2}^2 B_{3,2} B_{2,2} - \frac{10}{3} A_{2,2}^2 B_{3,2} B_{4,0} + \frac{82}{3} A_{2,2}^2 A_{3,1}^2 B_{4,0} + \frac{193}{6} A_{2,2}^2 A_{3,1} B_{3,1} + \frac{41}{30} A_{2,2}^2 A_{4,0} B_{3,1} - \frac{109}{5} A_{2,2} A_{3,1} A_{4,0}
\]

\[
+ \frac{5}{27} A_{2,2}^2 B_{4,0}^2 + \frac{127}{27} A_{3,1} A_{4,0}^2 - \frac{98}{27} A_{3,1} A_{4,0}^2 A_{4,0} + \frac{6}{5} A_{2,2} A_{4,0}^2 B_{4,0} - \frac{16}{5} A_{2,2} B_{3,1} B_{4,0} + \frac{2}{27} A_{2,2}^2 A_{4,0}^2 + \frac{17}{5} A_{2,2} B_{3,1} B_{4,0} - 6547 / 1650
\]

\[
A_{2,2}^2 B_{3,1}^2 + \frac{32}{9} B_{3,1}^2 B_{2,2} - \frac{35}{1728} A_{2,2}^2 B_{3,1} + \frac{53}{18} A_{2,2}^2 B_{2,2} B_{4,0} + 35A_{2,2}^2 A_{2,2} B_{3,1} + 2A_{2,2} A_{3,1} B_{2,2} - \frac{233}{6} A_{2,2}^2 B_{2,2} B_{4,0}
\]

\[
- \frac{3}{5} A_{2,2} A_{4,0}^2 B_{4,0} - \frac{7}{10} A_{3,1} A_{4,0} B_{3,1} + \frac{2}{3} A_{3,1} A_{4,0} B_{4,0} - \frac{5}{2} A_{2,2} A_{3,1} B_{3,1} + \frac{1}{2} A_{2,2} A_{4,0} B_{3,1} - \frac{58}{15} A_{4,0} B_{3,1} B_{4,0} - 45A_{2,2} B_{2,2} B_{4,0} - \frac{53}{15} A_{4,0}
\]
In the above expression of $\mu(k)$, we have already let $\mu(1) = \mu(2) = \ldots = \mu(k-1) = 0$, $k = 6$.

**Theorem 3.2:** In system (6), the first eighth singular point quantities at infinity are zero if and only if the following conditions hold

\begin{align}
\text{i.} & \quad A_{1,3}B_{4,0} \neq 0, A_{1,3} = 4B_{4,0}, \\
\text{ii.} & \quad A_{1,2}B_{3,0} \neq 0, 2A_{1,2} = 3B_{3,1}, \\
\text{iii.} & \quad A_{1,3}B_{2,2} \neq 0, 3A_{1,3} = 2B_{2,2}, \\
\text{v.} & \quad A_{4,4}B_{1,3} \neq 0, 4A_{4,4} = B_{1,3}.
\end{align}

**Proof:** The sufficiency condition is evident as $A_{1,3}B_{4,0} \neq 0$, since $\mu(3)$ is not zero and depends on $C_{0,3}, C_{2,1}, C_{1,2}$, and $C_{3,0}$, if all of them are zero then $\mu(3)$ is zero, and as a result the above conditions hold.

**Theorem 3.3** The infinity of system (2) is centre if and only if the four conditions in theorem 3.2 hold.

**Proof:** By the extended symmetric principle theorem (Theorem 4.3 in [7]) and the system satisfies all the conditions in theorem 3.2, then it is found that all the singular points of the origin are zero, and if all the singular points of the origin are zero, then this means that the origin is centre.

From the first eight singular points, the following theorem is obtained.
Theorem 3.4: The infinity of system (6) is a sixth fine singular point or infinity of system (2) is a sixth weak focus (i.e. \( \mu(1) = \mu(2) = \mu(3) = \mu(4) = \mu(5) = 0, \mu(6) \neq 0 \)) if and only if \( A_{2,2} = B_{2,2} = 0, A_{1,3} = B_{1,3} = 0, B_{3,1} = 0 A_{4,0} B_{4,0} \neq 0, A_{0,4} B_{0,4} \neq 0 \).

Proof: The proof is trivial by using the above theorem 3.3 and the Theorem 3.2 in [7].

4. Example for Bifurcation of Limit Cycles at Infinity

Now consider bifurcation of limit cycles at the origin of system (6). Being a polynomial differential system, the solution of (4) becomes

\[
v_{2m+1}(2\pi) = i\pi(\mu(m) + \sum_{k=1}^{m-1} \xi_m^{(k)} \mu(k)),
\]

where \( \xi_m^{(k)} (k = 1, 2, ..., m - 1) \), a polynomial function of the coefficients of system (6).

Theorem 4.1: If the coefficients of (6) satisfy the following

\[
A_{1,3} = B_{1,3} = 1, A_{2,2} = B_{2,2} = 0, A_{3,1} = 2\varepsilon, B_{3,1} = \frac{3}{2} \varepsilon, A_{0,4} = 3\varepsilon^3, A_{4,0} = \frac{3}{2} \varepsilon^3,
\]

\[
B_{0,4} = 9\varepsilon^2, B_{4,0} = \frac{9}{2} \varepsilon^2,
\]

and when \( \varepsilon = 0 \), then \( v_7(2\pi) \) tends to zero and there is no limit cycles and the system will have five weak fine foci. When \( 0 < \varepsilon << 1 \), there exists one limit cycle in a small enough neighbourhood of the origin of system (6).

Proof: According to quantities \( \mu(m) \), \( v_{2m+1}(2\pi) \approx i\pi\mu(m) \) and equation (10), the following computation is obtained

\[
v_1(2\pi) - 1 = \varepsilon - 1,
\]

\[
v_5(2\pi) = v_4(2\pi) = v_4(2\pi) = v_{14}(2\pi) = 0,
\]

\[
v_3(2\pi) = -\frac{3}{2} \varepsilon,
\]
\[ v_{13}(2\pi) = -0.0202 - 6.416e^2 + 649.18e^3 + 640e^4 + 439.09e^5 + 9.16e^6 - 2101.35e^7 - 3478.43e^8 - 205e^{10} + 13.8375e^{11}, \]

because the focal value of origin of system (6) will reverse its sign for only one instance, and from Theorem 6.6 in [7], there exists one limit cycle in a small enough neighbourhood of the origin of system (6). Correspondingly, there exists one limit cycle in a small enough neighbourhood of the infinity of system (2).

5. Conclusion

In the present work, the bifurcation of limit cycles at infinity for a class of homogeneous polynomial system of degree four is examined. The problem for bifurcation of limit cycles from infinity is transformed from the origin into the class of complex autonomous differential system. By calculation of singular point values, the conditions of the origin to be centre and the highest degree fine focus are obtained. Finally the quartic system is constructed, where it can bifurcate one limit cycle from infinity, when the normal parameters are constant values.

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6. Appendix

The computation of the singular point quantities at infinity of system (6)

\[ C_{0,0} = 0, \quad C_{0,1} = 0, \quad C_{0,2} = 0, \quad C_{0,3} = 0, \]
\[ C_{0,4} = 0, \quad C_{0,5} = 0, \quad C_{0,6} = 0, \quad C_{1,0} = 0, \quad C_{1,1} = 0, \quad C_{1,2} = 0, \quad C_{1,3} = 0, \quad C_{1,4} = 0, \quad C_{1,5} = 0, \quad C_{1,6} = 0, \]
\[ C_{2,0} = 0, \quad C_{2,1} = 2A_{2,2} - 3B_{2,1}, \quad C_{2,2} = -3A_{2,1} + 2B_{2,2}, \quad C_{2,3} = 1/3(-4A_{4,0} + B_{3,1}), \]
\[ C_{2,4} = 1/4[2/3A_{3,2}B_{3,1} - 8/3A_{2,3}B_{3,0} + 8B_{3,1}B_{3,0} - 6A_{1,3}A_{3,1} + 18A_{1,3}B_{2,0} + 4A_{3,3}B_{2,2} - 12B_{4,0}B_{3,2} - 8/3A_{4,0}A_{3,0} + 2/3A_{4,0}B_{3,0}], \]
\[ C_{2,5} = 1/10[10A_{2,1}A_{3,2} + 10/3B_{2,2}B_{1,0} + 6A_{2,2}^2 + 19A_{2,2}B_{1,1} + 15B_{2,1} + 11A_{3,1}B_{2,0} + 13/3A_{3,1}B_{2,2} - 4A_{2,1}A_{4,0} + A_{0,4}B_{4,1}], \]
\[ C_{2,6} = -1/2[5A_{2,1}B_{4,0} + 50/3A_{2,2}B_{2,2} - 13/3A_{2,2}B_{1,3}], \]
\[ C_{3,1} = 1/16[4A_{2,2}B_{4,0} + 6B_{3,1}B_{4,0} - 16A_{3,1}A_{2,1} + 6A_{3,1}B_{4,1} + 12A_{4,0}B_{2,2} - 4B_{1,3}B_{2,2} - 8A_{1,3}A_{3,1} + 8/3A_{3,2}B_{2,2} - 2/3A_{4,0}B_{2,2}], \]
\[ C_{3,4} = -1/6[4A_{2,2}B_{4,0} - 2B_{4,0}B_{2,2} - 28/3A_{2,2}B_{1,3} + 11/3A_{4,0}B_{1,3} - 1/3B_{2,2}], \]
\[ C_{4,0} = 0, \quad C_{4,1} = 0, \quad C_{4,2} = 0, \quad C_{4,3} = 0, \quad C_{4,4} = 0, \]
\[ C_{4,5} = 0, \quad C_{4,6} = 0, \quad C_{4,7} = 1/5[3A_{4,1} - 8B_{2,2} + (3A_{2,2} - 8B_{1,3})C_{4,6} + (3A_{3,3} - 8B_{4,0})C_{4,2}]. \]
\( C_{3,6} = \frac{1}{3} [(4A_{4,0} - 7B_{2,2})C_{0,6} + (4A_{3,1} - 7B_{2,2})C_{1,5} + (4A_{2,2} - 7B_{3,1})C_{2,4} + 4A_{2,2}C_{4,2}] \),
\( C_{4,5} = -6B_{0,4}C_{0,6} + (5A_{4,0} - 6B_{1,3})C_{1,5} + (5A_{3,1} - 6B_{2,2})C_{2,4} + (5A_{2,2} - 6B_{3,1})C_{4,2} + 5A_{0,4}C_{5,1} \),
\( C_{5,4} = -[5B_{0,4}C_{1,5} + (6A_{4,0} - 5B_{1,3})C_{2,4} + (6A_{3,1} - 5B_{2,2})C_{4,2} + (6A_{2,2} - 5B_{3,1})C_{5,1} + 6A_{0,4}C_{6,0}] \),
\( C_{6,3} = -\frac{1}{3} [-4B_{0,4}C_{2,4} + (7A_{3,1} - 4B_{2,2})C_{4,2} + (7A_{2,2} - 4B_{3,1})C_{4,2} + (7A_{2,2} - 4B_{3,1})C_{6,0}] \),
\( C_{7,2} = -\frac{1}{5} [(8A_{4,0} - 3B_{1,3})C_{4,2} + (8A_{3,1} - 3B_{2,2})C_{5,1} + (8A_{2,2} - 3B_{3,1})C_{6,0}] \).

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