A New Class of k-Equitable Trees

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Abstract

Cahit introduced k-equitable labeling as a generalization of graceful labeling. For any graph $G(V,E)$ and any positive integer $k$, a function $f$ defined from the vertex set of $G$ to $\{0,1,2,\ldots,k-1\}$ is called k-equitable if every edge $uv$ is assigned the label $|f(u) - f(v)|$, then the number of vertices labeled $i$ and the number of vertices labeled $j$ differ by at most 1 and the number of edges labeled $i$ and the number of edges labeled $j$ differ by at most 1, for $0 \leq i < j \leq k-1$. In this paper we show that a class of bamboo trees are k-equitable.

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1 Introduction

At the Smolenice Symposium in 1963, Ringel [6] conjectured that $K_{2m+1}$, the complete graph on $2m+1$ vertices can be decomposed into $2m+1$ isomorphic copies of a given tree with $m$ edges. In an attempt to solve Ringel’s conjecture,
in 1967 Rosa [7] introduced an hierarchical series of labelings called $\rho, \sigma, \beta$ and $\alpha$-valuations and used these valuations for a cyclic decomposition of $K_{2m+1}$ into trees with $m$ edges. Later Golomb [4] called $\beta$-valuation as graceful.

A function $f$ is called a graceful labeling of $G$ with $m$ edges, if $f$ is an injection from the vertices of $G$ to the set $\{0, 1, 2, \ldots, m\}$ such that when each edge $uv$ is assigned the label $|f(u) - f(v)|$ then the resulting edge labels are distinct.

In 1990, Cahit [3] proposed the idea of distributing the vertex and edge labels among $\{0, 1, 2, \ldots, k-1\}$ as evenly as possible to obtain a generalization of graceful labeling as follows.

A vertex labeling of a graph $G = (V, E)$ is a function $f : V(G) \rightarrow \{0, 1, 2, \ldots, k-1\}$ and the value of $f(u)$ is called the label of the vertex $u$. For the vertex labeling function $f : V(G) \rightarrow \{0, 1, 2, \ldots, k-1\}$, the induced edge function $f^* : E(G) \rightarrow \{0, 1, 2, \ldots, k-1\}$ is defined as $f^*(e = uv) = |f(u) - f(v)|$. Such a labeling $f$ is called $k$-equitable labeling of $G$ if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $0 \leq i < j \leq k-1$, where $v_f(i)$ and $e_f(i)$ denote the number of vertices and the number of edges having label $i$ under $f$ and $f^*$ respectively. A graph which admits $k$-equitable labeling is called $k$-equitable graph. Cahit [1] proved that every tree is 2-equitable. Speyer and Szaniszlo [9] proved that every tree is 3-equitable. Szaniszlo [8] proved that every path is k-equitable and every star is $k$-equitable. For an exhaustive survey on $k$-equitable graph refer the excellent dynamic survey by Gallian [5].

In 1990, Cahit [2] conjectured that every tree is $k$-equitable for any $k \geq 2$. This conjecture is equivalent to the celebrated graceful tree conjecture when $k$ is the number of vertices of the tree. One possible approach to prove the popular graceful tree conjecture is to prove the more general $k$-equitable tree conjecture. Inspired by this general approach, in this paper we show that a new class of trees called bamboo trees are $k$-equitable for $k \geq 3$, where the bamboo tree is defined below.

A tree is called bamboo tree if it is obtained from $r$ stars $S_{n_1}, S_{n_2}, \ldots, S_{n_r}$, where the size $n_i$ of each star $S_{n_i}$, $1 \leq i \leq r$ is an arbitrary positive integer, by joining the center of each of the stars $S_{n_i}$ to a new vertex $v$ called the root, by a path $P_i$ of length $\ell_i$ (where the length $\ell_i$ may vary for each path $P_i$), $1 \leq i \leq r$ and it is denoted by $B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)$.

## 2 Main Result

In this section we prove our main result in Theorem 2.1.

**Theorem 2.1.** For $k \geq 3$, the bamboo tree $B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)$ with $\ell_i = m_i k$, for $i$, $1 \leq i \leq r$, such that either all the $m_i$’s are even positive
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Integers or all the $m_i$'s are odd positive integers is $k$-equitable.

Proof. Consider the bamboo tree $B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)$ with $\ell_i = m_i k$ such that either all the $m_i$'s are even positive integers or all the $m_i$'s are odd positive integers, for $i$, $1 \leq i \leq r$.

By the definition, $B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)$ is obtained from $r$ stars $S_{n_1}, S_{n_2}, \ldots, S_{n_r}$, where $n_i$, the size of the star $S_{n_i}$, is an arbitrary positive integer, for $i$, $1 \leq i \leq r$ and the center of each star $S_{n_i}$ is joined to the root $v$ by a path $P_i$ of length $\ell_i = m_i k$. Thus $B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)$ has $r$ branches at the root vertex $v$ and each $i^{th}$ branch is of depth $\ell_i + 1 = m_i k + 1$ having $n_i \geq 1$ leaves for $1 \leq i \leq r$.

To label the vertices of $B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)$ first we label the vertices of the path $P_i$ in the $i^{th}$ branch for each $i$, $1 \leq i \leq r$. Then we label all the leaves of $B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)$.

The $i^{th}$ branch of $B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)$ is shown in Figure 1.

![Figure 1: i^{th} branch of B(n_1, n_2, ..., n_r, \ell_1, \ell_2, ..., \ell_r)](image)

**Step 1.** Labeling of the vertices of the path $P_i$ of the $i^{th}$ branch, for $i$, $1 \leq i \leq r$.

We label the vertices of the path $P_i$ in two cases depending on all the $m_i$'s are even positive integers or all the $m_i$'s are odd positive integers, where $m_i k$ is the length of the path $P_i$, for $i$, $1 \leq i \leq r$.

**Case 1.** All the $m_i$'s are even positive integers, for $i$, $1 \leq i \leq r$, where $m_i k$ is the length of the path $P_i$. 
Assign \((k-1)\) to the root of \(B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)\). For \(i, 1 \leq i \leq r\), consider the path \(P_i - v\) of the \(i^{th}\) branch of \(B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)\). Observe that the path \(P_i - v\) has \(\ell_i = m_ik\) vertices. Since \(m_i\) is even we can write \(\ell_i = \left(\frac{m_i}{2}\right)2k\). Thus, we consider the path \(P_i - v\) as union of \(\frac{m_i}{2}\) subpaths \(P_{is}\) and each subpath \(P_{is}\) has \(2k\) vertices, for \(s, 1 \leq s \leq \frac{m_i}{2}\) as shown in Figure 2.

Figure 2: The subpath decomposition \(P_{is}\), for \(s, 1 \leq s \leq \frac{m_i}{2}\), of the path \(P_i\) of length \(\ell_i = m_i k\) in the \(i^{th}\) branch when \(m_i\) is even.

Now for each \(s, 1 \leq s \leq \frac{m_i}{2}\), we label the \(2k\) vertices of the subpath \(P_{is}\) as shown in Figure 3(a) and Figure 3(b).
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Figure 3: (a) Labeling of the $2k$ vertices of the subpath $P_{is}$, for $s, 1 \leq s \leq \frac{m}{2}$ of the path $P_i$ of length $\ell_i = m_i k$ when $k$ is even

Figure 3(b) Labeling of the $2k$ vertices of the subpath $P_{is}$, for $s, 1 \leq s \leq \frac{m}{2}$ of the path $P_i$ of length $\ell_i = m_i k$ when $k$ is odd.

From Figure 3(a) and Figure 3(b), we observe that each of the labels $0, 1, \ldots, k - 1$ is assigned to exactly two vertices of $P_{is}$ and each of the labels $1, 2, \ldots, k - 1$ is obtained exactly at two edges of $P_{is}$, and the label zero is obtained in the middle edge of $P_{is}$, for $s, 1 \leq s \leq \frac{m}{2}$.

Also observe that the edge connecting the subpaths $P_{is}$ and $P_{is+1}$, for $s, 1 \leq s \leq (\frac{m}{2}) - 1$ always gets the label 0.

Thus the path $P_i - v$ contains $m_i$ sets of $k$ vertices with labels $0, 1, 2, \ldots, k - 1$ and consequently $m_i k$ edges of the path $P_i$ gets $m_i$ sets of the labels $0, 1, 2, \ldots, k - 1$.

**Case 2.** All the $m_i$’s are odd positive integers, for $i, 1 \leq i \leq r$, where $m_i k$ is the length of the path $P_i$. 
Figure 4: The subpath decomposition $P_{i,s}$, for $s, 1 ≤ s ≤ \frac{m_i+1}{2}$ of the path $P_i$ of length $\ell_i = m_i k$ in the $i^{th}$ branch when $m_i$ is odd.
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Figure 5: (a) For the case $m_i$ is odd, labeling of the $k$ vertices of the last subpath $P_{i\frac{m_i+1}{2}}$ of the path $P_i$ of length $\ell_i = m_i k$ in the $i^{th}$ branch when $k$ is even

Figure 5(b) For the case $m_i$ is odd, labeling of the $k$ vertices of the last subpath $P_{i\frac{m_i+1}{2}}$ of the path $P_i$ of length $\ell_i = m_i k$ in the $i^{th}$ branch when $k$ is odd

Assign $\left\lfloor \frac{k-1}{2} \right\rfloor$ to the root of $B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)$. For $i, 1 \leq i \leq r$, consider the path $P_{i} - v$ of the $i^{th}$ branch of $B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)$. Observe that the path $P_{i} - v$ has $\ell_i = m_i k$ vertices. Since $m_i$ is odd we can write $\ell_i = (\frac{m_i-1}{2}) 2k + k$. Thus, we consider the path $P_{i} - v$ as union of $\frac{m_i-1}{2}$ subpaths $P_{i_s}$, where each subpath $P_{i_s}$ has $2k$ vertices, for $s, 1 \leq s \leq \frac{m_i-1}{2}$ and the last subpath $P_{i\frac{m_i+1}{2}}$ has $k$ vertices as shown in Figure 4.

For $s, 1 \leq s \leq \frac{m_i-1}{2}$, we give the labels to the $2k$ vertices of the subpath $P_{i_s}$ as labeled in Figure 3(a) and Figure 3(b). Then we label the $k$ vertices of the last subpath $P_{i\frac{m_i+1}{2}}$ as shown in Figure 5(a) and Figure 5(b).

From Figure 5(a) and Figure 5(b) we observe that each of the labels $0, 1, 2, \ldots, k - 1$ is assigned exactly once to the vertices of the last subpath $P_{i\frac{m_i+1}{2}}$ and each of the labels $1, 2, \ldots, k - 1$ is obtained exactly once to the edges of $P_{i\frac{m_i+1}{2}}$ and the edge label $0$ is obtained at the edge connecting the root and the subpath $P_{i\frac{m_i+1}{2}}$.

It follows from the above observation and from Case 1, the path $P_{i} - v$ contains $m_i$ set of $k$ vertices with labels $0, 1, 2, \ldots, k - 1$ and consequently $m_i k$ edges of the path $P_i$ gets $m_i$ sets of the labels $0, 1, 2, \ldots, k - 1$. 

\textbf{Figure 5:} (a) For the case $m_i$ is odd, labeling of the $k$ vertices of the last subpath $P_{i\frac{m_i+1}{2}}$ of the path $P_i$ of length $\ell_i = m_i k$ in the $i^{th}$ branch when $k$ is even

\textbf{Figure 5(b) For the case $m_i$ is odd, labeling of the $k$ vertices of the last subpath $P_{i\frac{m_i+1}{2}}$ of the path $P_i$ of length $\ell_i = m_i k$ in the $i^{th}$ branch when $k$ is odd}
Step 2. Labeling of all the leaves of $B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)$

Let $A$ be the set of all leaves of $B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)$. Then observe that $|A| = n_1 + n_2 + \cdots + n_r = t$. Let $t = ak + b$, where $a$ and $b$ are positive integers and $0 \leq b \leq k - 1$. For the convenience we consider the $t$ leaves of $B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)$ as an ordered set of $k$ leaves and one more set of $b$ leaves from left to right. For $i$, $1 \leq i \leq a$, let $A_i$ denote the $i^{th}$ set of $k$ leaves from left to right and $A_{a+1}$ denote the last set of $b$ leaves. For each $i$, $1 \leq i \leq a$ label the $k$ leaves of $A_i$ as $0, 1, 2, \ldots, k - 1$.

Finally, to label the $b$ leaves of $A_{a+1}$ we consider two cases,

Case I. If the label of the root is $k - 1$

Then label the $b$ leaves of $A_{a+1}$ as $0, 1, 2, \ldots, b - 1$.

From Case I, we observe that the number of leaves labeled with $i$ and the number of leaves labeled with $j$ are equal if $i, j \in \{0, 1, 2, \ldots, b - 1\}$ or $i, j \in \{b, \ldots, k - 1\}$ and the number of leaves labeled with $i$ and the number of leaves labeled with $j$ differ by at most 1 if $i \in \{0, 1, 2, \ldots, b - 1\}$ and $j \in \{b, \ldots, k - 1\}$.

Also observe that the number of pendant edges (the edges incident with the center of the stars) labeled with $i$ and the number of pendant edges labeled with $j$ are equal if $i, j \in \{0, 1, 2, \ldots, b - 1\}$ or $i, j \in \{b, \ldots, k - 1\}$ and the number of pendant edges labeled with $i$ and the number of pendant edges labeled with $j$ differ by at most 1 if $i \in \{0, 1, 2, \ldots, b - 1\}$ and $j \in \{b, \ldots, k - 1\}$.

Case II. If the label of the root is $\lfloor \frac{k-1}{2} \rfloor$

Then we label the $b$ leaves in $A_{a+1}$ as $\{0, \ldots, \lfloor \frac{k-1}{2} \rfloor - 1, \lfloor \frac{k-1}{2} \rfloor + 1, \ldots, b\}$ if $\lfloor \frac{k-1}{2} \rfloor < b$ or else $\{0, 1, \ldots, b - 1\}$ if $\lfloor \frac{k-1}{2} \rfloor \geq b$.

Case IIa. When $\lfloor \frac{k-1}{2} \rfloor < b$.

Then we have the number of leaves labeled with $i$ and the number of leaves labeled with $j$ are equal if $i, j \in \{0, \ldots, \lfloor \frac{k-1}{2} \rfloor - 1, \lfloor \frac{k-1}{2} \rfloor + 1, \ldots, b\}$ or $i, j \in \{\lfloor \frac{k-1}{2} \rfloor, b + 1, \ldots, k - 1\}$ and the number of leaves labeled with $i$ and the number of leaves labeled with $j$ differ by at most 1 if $i \in \{0, \ldots, \lfloor \frac{k-1}{2} \rfloor - 1, \lfloor \frac{k-1}{2} \rfloor + 1, \ldots, b\}$ and $j \in \{\lfloor \frac{k-1}{2} \rfloor, b + 1, \ldots, k - 1\}$.

The induced edge labels of the pendant edges will depend on $k$ is odd or $k$ is even.

When $k$ is odd.

Then the number of pendant edges labeled with $i$ and the number of pendant edges labeled with $j$ are equal if $i, j \in \{k - 1, k - 2, \ldots, \frac{k+1}{2}, \frac{k-1}{2}, \ldots, \}$,
\(k - b - 1\) or \(i, j \in \{ \frac{k-1}{2}, k - b - 2, k - b - 3, \ldots, 0\}\) and the number of pendant edges labeled with \(i\) and the number of pendant edges labeled with \(j\) differ by at most 1 if \(i \in \{ k - 1, k - 2, \ldots, \frac{k+1}{2}, \frac{k-2}{2}, \ldots, k - b - 1\}\) and \(j \in \{ \frac{k-1}{2}, k - b - 2, k - b - 3, \ldots, 0\}\).

**When \(k\) is even.**

Then the number of pendant edges labeled with \(i\) and the number of pendant edges labeled with \(j\) are equal if \(i, j \in \{ k - 1, k - 2, \ldots, \frac{k+1}{2}, \frac{k-2}{2}, \ldots, k - b - 1\}\) or \(i, j \in \{ \frac{k}{2}, k - b - 2, k - b - 3, \ldots, 0\}\) and the number of pendant edges labeled with \(i\) and the number of pendant edges labeled with \(j\) differ by at most 1 if \(i \in \{ k - 1, k - 2, \ldots, \frac{k+2}{2}, \frac{k-2}{2}, \ldots, k - b - 1\}\) and \(j \in \{ \frac{k}{2}, k - b - 2, k - b - 3, \ldots, 0\}\).

**Case IIb.** When \(\left[ \frac{k-1}{2} \right] \geq b\).

Then we have the number of leaves labeled with \(i\) and the number of leaves labeled with \(j\) are equal if \(i, j \in \{ 0, 1, \ldots, b - 1\}\) or \(i, j \in \{ b, \ldots, k - 1\}\) and the number of leaves labeled with \(i\) and the number of leaves labeled with \(j\) differ by at most 1 if \(i \in \{ 0, 1, \ldots, b - 1\}\) and \(j \in \{ b, \ldots, k - 1\}\).

Similarly the number of pendant edges labeled with \(i\) and the number of pendant edges labeled with \(j\) are equal if \(i, j \in \{ k - 1, k - 2, \ldots, k - b\}\) or \(i, j \in \{ k - b - 1, k - b - 2, \ldots, 0\}\) and the number of pendant edges labeled with \(i\) and the number of pendant edges labeled with \(j\) differ by at most 1 if \(i \in \{ k - 1, k - 2, \ldots, k - b\}\) and \(j \in \{ k - b - 1, k - b - 2, \ldots, 0\}\).

From Step 1 and Step 2 we observe that the number of vertices of \(B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)\) labeled with \(i\) and the number of vertices labeled with \(j\) are equal or differ by at most 1. Similarly the number of edges of \(B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)\) labeled with \(i\) and the number of edges labeled with \(j\) are equal or differ by at most 1.

Hence the bamboo tree \(B(n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_r)\) is \(k\)-equitable. \(\square\)

**References**


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