Theory of Generalized $k$-Difference Operator and Its Application in Number Theory

V. Chandrasekar

Department of Mathematics, S.K.P. Engineering College
Tiruvannamalai - 606 611, Tamil Nadu, S. India

G. Britto Antony Xavier

Department of Mathematics, Sacred Heart College
Tirupattur - 635 601, Tamil Nadu, S. India

R. Vijayaraj

Department of Mathematics, Arunai Engineering College
Tiruvannamalai - 606 603, Tamil Nadu, S. India

Abstract

In this paper, we define the generalized $k$-difference operator and present the discrete version of Leibnitz theorem according to the generalized $k$-difference operator. Also, by defining its inverse, we obtain the sum of product of arithmetic and geometric progressions in the field of Numerical Analysis.

Mathematics Subject Classification: 39A70, 47B39, 97N40

Keywords: $k$-Difference Operator, Generalized Factorial, Polynomial Factorial
1 Introduction

The theory of Difference equations is based on the operator $\Delta$ defined as

$$\Delta u(k) = u(k + 1) - u(k), \; k \in \mathbb{N} = \{0, 1, 2, \ldots\}. \hspace{1cm} (1)$$

Even though many authors [1, 7–9] have suggested the definition of the difference operator $\Delta$ as

$$\Delta u(k) = u(k + \ell) - u(k), \; \ell \in (0, \infty), \hspace{1cm} (2)$$

no significant progress took place on this line. But, in 2006 [3], by taking the definition of $\Delta$ as given in (2), the theory of difference equations was developed in a different direction and many interesting results were obtained in Number Theory [4].

Jerzy Popenda [2], while discussing the behavior of solutions of particular type of difference equation, defined $\Delta_{\alpha}$ as

$$\Delta_{\alpha} u(k) = u(k + 1) - \alpha u(k). \hspace{1cm} (3)$$

This definition of $\Delta_{\alpha}$ is being ignored for a long time. But, recently, M.M.S.Manuel, V.Chandrasekar and G.Britto Antony Xavier [5, 6], have generalized the definition of $\Delta_{\alpha}$ by $\Delta_{\alpha(\ell)}$ defined as

$$\Delta_{\alpha(\ell)} u(k) = u(k + \ell) - \alpha u(k) \hspace{1cm} (4)$$

for the real valued function $u(k)$ and $\ell \in (0, \infty)$ and also obtained the solutions of certain type of generalized $\alpha$-difference equations, in particular generalized clairaut’s $\alpha$-difference equation, generalized Euler $\alpha$-difference equation and formula for the sum of several types of arithmetic-geometric progressions in Number Theory using the generalized $\alpha$-Bernoulli polynomial, which is a solution of the generalized $\alpha$-difference equation as $u(k + \ell) - \alpha u(k) = nk^{n-1}$, for $n \in \mathbb{N}(1)$. 

With this background, in this paper, we define the generalized $k$-difference operator and obtain some results, relations and theorems. Also, we derive the formula for sum of product of factorials and arithmetic progressions in the field of Numerical Analysis using the inverse of generalized $k$-difference operator.

Throughout this paper, the following assumptions are used:

(i) $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, \hspace{1cm} (ii) $\mathbb{N}_\ell(j) = \{j, j + \ell, j + 2\ell, \ldots\}$,

(iii) $$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$, \hspace{1cm} (iv) $\Delta^{-1}_{k(\ell)} u(k)|^k_j = \Delta^{-1}_{k(\ell)} u(k) - (k - \ell)_{\ell}^{\left(\binom{j}{\ell}\right)} \Delta^{-1}_{k(\ell)} u(j)$,

(v) $\left\lfloor \frac{k}{\ell} \right\rfloor$ is the integer part of $\frac{k}{\ell}$ and $\left\lceil \frac{k}{\ell} \right\rceil$ is the upper integer part of $\frac{k}{\ell}$. 

2 Generalized $k$-Difference Operator and its Relation with $\Delta$ and $E$

In this section, we present some basic definitions and preliminary results which will be useful for further subsequent discussions.

**Definition 2.1** Let $u(k)$ be a real valued function defined on $[0, \infty)$. Then the generalized $k$-difference operator $\Delta_{k(\ell)}$ is defined as

$$\Delta_{k(\ell)} u(k) = u(k + \ell) - ku(k), \quad \ell \in (0, \infty), \quad k \in [0, \infty). \quad (5)$$

**Remark 2.2** (i) If $\alpha$ is a constant, then the generalized $\alpha$-difference operator $\Delta_{\alpha(\ell)}$ assumes the relation $\Delta_{\alpha(\ell)} u(k) = u(k + \ell) - \alpha u(k)$.

(ii) Replacing $k$ by $k \pm n\ell$ in (5), we obtain $\Delta_{k(\ell)} u(k \pm n\ell) = u(k \pm n\ell + \ell) - ku(k \pm n\ell)$, for appropriate choosing $n \in \mathbb{N}$.

The following lemma is the relation between $\Delta_{k(\ell)}$, $\Delta_\ell$ and $E$.

**Lemma 2.3** (i) If $E_\ell$ is the usual shift operator, then

$$E^\ell u(k) = (\Delta_{k(\ell)} + k)u(k) = (1 + \Delta)^\ell u(k) = (1 + \Delta_\ell)u(k). \quad (6)$$

(ii) If $m$ is positive integer, then

$$(k + \Delta_{k(m)})u(k) = \sum_{i=0}^{m} \binom{m}{i} \Delta^i u(k). \quad (7)$$

(iii) If $c_1$, $c_2$ are any two non-zero scalars and $u(k)$, $v(k)$ are any two real valued functions defined on $k \in [0, \infty)$, then

$$\Delta_{k(\ell)}[c_1 u(k) + c_2 v(k)] = c_1 \Delta_{k(\ell)} u(k) + c_2 \Delta_{k(\ell)} v(k).$$

(iv) If $\ell \in (0, \infty)$ and $n$ is a positive integer, then

$$\Delta_{k(\ell)}^n u(k) = \sum_{r=0}^{n} \binom{n}{r} (-k)^r u(k + \ell(n - r)). \quad (8)$$

(v) If $\ell_i$, $i = 1, 2, \ldots n$ are positive reals, then

$$(k + \Delta_{k(\ell_1 + \ell_2 + \ell_3 + \ldots + \ell_n)}) u(k) = \prod_{i=1}^{n} (k + \Delta_{k(\ell_i)}) u(k). \quad (9)$$

(vi) If $n$ is a positive integer, then

$$\Delta_{k(\ell^n)} u(k) = ((1 + \Delta)^n - k) u(k). \quad (10)$$
The following is the discrete version of Leibnitz theorem according to $\Delta_{k(\ell)}$.

**Theorem 2.4** If $u(k)$ and $v(k)$, $k \in [0, \infty)$, are any two real valued functions, then

$$
\Delta_{k(\ell)}^n [u(k)v(k)] = \sum_{r=0}^{n} \binom{n}{r} k^{n-r} (\Delta_{k(\ell)}^r u(k)) (\Delta_{k(\ell)}^{n-r} v(k + r\ell)).
$$

**Proof.** Define the shift operator $E_1^\ell$ and $E_2^\ell$ as

$$
E_1^\ell [u(k)v(k)] = u(k + \ell)v(k) \text{ and } E_2^\ell [u(k)v(k)] = u(k)v(k + \ell).
$$

(11)

Hence, we obtain

$$
E^\ell = E_1^\ell E_2^\ell.
$$

(12)

Let us denote

$$
(\Delta_{k(\ell)})_1 = E_1^\ell - k \text{ and } (\Delta_{k(\ell)})_2 = E_2^\ell - k
$$

(13)

for the functions $u(k)$ and $v(k)$, respectively.

This implies

$$
\Delta_{k(\ell)} = E_1^\ell E_2^\ell - k.
$$

From (6), we obtain

$$
\Delta_{k(\ell)} = (\Delta_{k(\ell)})_1 E_2^\ell - k (\Delta_{k(\ell)})_2,
$$

where $E_2^\ell - 1 = (\Delta_{k(\ell)})_2$ ([2], Theorem 2.5). Hence,

$$
\Delta_{k(\ell)}^n [u(k)v(k)] = [k(\Delta_{k(\ell)})_2 + (\Delta_{k(\ell)})_1] E_2^\ell^n [u(k)v(k)].
$$

(14)

The proof follows by Binomial Theorem, (11), (13) and (14).

**Lemma 2.5** If $\ell \in (0, \infty)$ and $n$ is a positive integer, then

$$
E^{n\ell} u(k) = \sum_{r=0}^{n} \binom{n}{r} k^{n-r} \Delta_{k(\ell)}^r u(k).
$$

(15)

**Proof.** The proof follows from (6) and Binomial Theorem.

The following is discrete version of generalized Binomial theorem involving $\Delta_{k(\ell)}$.

**Theorem 2.6** If $m$ and $n$ are any two positive integers, then

$$
(k + n\ell)^m = \sum_{r=0}^{n} \sum_{t=0}^{r} (-1)^{n-r} \binom{n}{r} \binom{r}{t} k^{n+r-t} [k + \ell(n - t)]^m.
$$

(16)

**Proof.** The proof follows by taking $u(k) = k^m$ in (15) and using (8).
Lemma 2.7 Let \( u(k), k \in [0, \infty) \), be a real valued function and \( \ell \in (0, \infty) \).

Then
\[
\sum_{r=0}^{\infty} \frac{x^{r\ell} u(r\ell)}{r! \ell^r} = e^{x\ell} e^{\frac{\Delta_k \ell}{\ell}} u(0).
\] (17)

Proof. From the shift operator, \( E^{r\ell} u(0) = u(r\ell) \) and binomial theorem, we find
\[
e^{\frac{x\ell}{\ell}} u(0) = \sum_{r=0}^{\infty} \frac{x^{r\ell} u(r\ell)}{r! \ell^r}.
\] (18)

The proof follows from (6) and (18).

The following theorem is the generalized version of Montmorte’s theorem with reference to \( \Delta_{k(\ell)} \).

Theorem 2.8 If the series \( \sum_{k=0}^{\infty} u(k\ell)x^{k\ell}, \ k \in \mathbb{N} \) converges, then it can be expressed as
\[
\sum_{k=0}^{\infty} u(k\ell)x^{k\ell} = \sum_{k=0}^{\infty} \frac{x^{k\ell} \Delta_k \ell u(0)}{(1 - kx\ell)^{k+1}}.
\] (19)

Proof. From the shift operator \( E^\ell \), we have
\[
\sum_{k=0}^{\infty} u(k\ell)x^{k\ell} = \sum_{k=0}^{\infty} x^{k\ell} E^\ell u(0) = (1 - x\ell E^\ell)^{-1} u(0).
\]

(19) follows from (6) and the above relation.

3 The Generalized Factorial

In this section, we define the generalized factorial and obtain the relation between the generalized \( k \)-difference operator and generalized factorial.

Definition 3.1 If \( \ell \in (0, \infty) \) and \( k \in [\ell, \infty) \), then the Generalized Factorial denoted by \( (k_\ell)! \) is defined as
\[
(k_\ell)! = k(k - \ell)(k - 2\ell) \cdots j,
\] (20)

where \( j = k - \left\lfloor \frac{k}{\ell} \right\rfloor \ell \).

Theorem 3.2 Let \( \ell \in (0, \infty) \) and \( k \in [\ell, \infty) \). Then
\[
\Delta_{k(\ell)}(k_\ell)! = \ell(k_\ell)!
\] (21)

Proof. The proof follows from (5) and (20).
Lemma 3.3 Let $k^{(n)}_\ell = k(k - \ell)(k - 2\ell)...(k - (n - 1)\ell)$ be the generalized polynomial factorial. Then

$$\Delta k^{(n)}_\ell = k^{(n-1)}_\ell [k(1 - k) + (1 + (n - 1)k)\ell]. \quad (22)$$

Proof. The proof follows from (5) and $k^{(n-1)}_\ell = k(k - \ell)(k - 2\ell)...(k + 2\ell - n\ell)$.

Theorem 3.4 If $m$ and $n$ are any two positive integer, then

$$\Delta^m k^{\ell} \left( \left( \frac{k}{\ell} \right) \right)^n = \prod_{i=1}^{m} \sum_{t=1}^{n-i} \left( n + (m - i) - \sum_{t=1}^{r_t} \ell r_{m+1-i} \right) \ell r_{m+1-i} \prod_{p=1}^{m+1} k^{\ell+1} \left( \left( \frac{k}{\ell} \right) \right). \quad (23)$$

Proof. The proof follows by induction on $m$.

Corollary 3.5 If $n$ is any positive integer, then

$$\Delta^n k^{\ell} (k - \ell)^{\left( \left( \frac{k}{\ell} \right) \right)} = k^{\ell+1} \sum_{t=1}^{n} \left( \begin{array}{c} n \\ t \end{array} \right) \ell^n t^t. \quad (24)$$

Proof. The proof follows by substituting $m = 1$ in (23).

Note that when $k$ is multiple of $\ell$, $(k - \ell)^{\left( \left( \frac{k}{\ell} \right) \right)} = 0$ and $k^{\ell+1} = 0$, hence bothsides of (24) are zero.

4 Inverse of the Operator $\Delta_k(\ell)$

In this section, we define the inverse of the operator $\Delta_k(\ell)$ and obtaining some theorems involving the inverse of generalized $k$-difference operator.

Definition 4.1 The inverse of the generalized $k$-difference operator is denoted by $\Delta_k^{-1}(\ell)$ is defined as follows. If $\Delta_k(\ell) v(k) = u(k)$, then

$$\Delta_k^{-1}(\ell) u(k) = v(k) - (k - \ell)^{\left( \left( \frac{k}{\ell} \right) \right)} v(j), \quad (25)$$

where $v(j)$ is constant for all $k \in \mathbb{N}_\ell(j)$.

Theorem 4.2 If $\ell \in (0, \infty)$ and $k \in [\ell, \infty)$, then

$$\Delta_k^{-1}(\ell) u(k) \bigg|_j^k = \sum_{r=1}^{\left( \left( \frac{k}{\ell} \right) \right)} (k - \ell)^{r-1} u(k - r\ell). \quad (26)$$
Proof. The proof follows from Definition 4.1, and the relation
\[
\Delta_{k}(\ell) \left\{ \sum_{r=1}^{[\frac{k}{\ell}]} (k - \ell)^{r-1} u(k - r\ell) \right\} = u(k).
\]

**Theorem 4.3** Let \( \lambda \neq 1 \), \( k \geq 2\ell \) and \( p(k) \) be a function in \( k \). Then
\[
\sum_{r=1}^{[\frac{k}{\ell}]} (k - \ell)^{r-1} \lambda^{k-r\ell} P(k - r\ell)
= \left\{ \frac{\lambda^k}{k(\lambda^\ell - 1)} - \frac{\lambda^{k+\ell} \Delta_{k}(\ell)}{k^2(\lambda^\ell - 1)^2} + \frac{\lambda^{k+2\ell} \Delta^2_{k}(\ell)}{k^3(\lambda^\ell - 1)^3} + \cdots \right\} p(k)
= \lambda^k P(k) \bigg|_{j}^k.
\]

(27)

Proof. For the function \( F(k) \), we find
\[
\Delta_{k}(\ell) \lambda^k F(k) = \lambda^k (\lambda^\ell E^\ell - k) F(k) = \lambda^k P(k),
\]

(28)

hence, we obtain
\[
(\lambda^\ell E^\ell - k)^{-1} P(k) = F(k).
\]

(29)

From (28) and Definition 4.1, we find
\[
\Delta^{-1}_{k}(\ell) (\lambda^k P(k)) = \lambda^k F(k) - (k - \ell)_{[\frac{k}{\ell}]} \lambda^j F(j),
\]

and hence by (29), we obtain
\[
\Delta^{-1}_{k}(\ell) (\lambda^k P(k)) = \lambda^k (\lambda^\ell E^\ell - k)^{-1} P(k) - (k - \ell)_{[\frac{k}{\ell}]} \lambda^j (\lambda^\ell E^\ell - j)^{-1} P(j).
\]

(30)

The proof now follows from (26), (30) and Binomial theorem.

5 Applications

In section, we establish the formula for the sum of product of factorials and arithmetic progression and arithmetic geometric progression in number theory as an application of \( \Delta_{k}(\ell) \).

**Lemma 5.1** Let \( \ell \in (0, \infty) \) and \( k \in [\ell, \infty) \). Then
\[
\Delta^{-1}_{k}(\ell)! = \frac{(k\ell)!}{\ell}.
\]

(31)

Proof. The proof follows from (21) and Definition 4.1.
Theorem 5.2 Let \( \ell \in (0, \infty) \) and \( k \in [\ell, \infty) \). Then

\[
\left[ \frac{k}{\ell} \right] \sum_{r=1}^{k-\left\lceil \frac{k}{\ell} \right\rceil} (k - \ell)^{(r-1)} ((k - r\ell)_\ell)! = \frac{1}{\ell}[(k_\ell)! - (k - \ell)\left(\left[ \frac{k}{\ell} \right]\right)(j_\ell)!]. \tag{32}
\]

Proof. The proof follows by taking \( u(k) = (k_\ell)! \) in (26) and (31).

Example 5.3 Substituting \( k = 70, \ell = 3 \) and \( j = 1 \) in (32), we get

\[
(67)_3! + (67)(1)(64)_3! + (67)(2)(61)_3! + \cdots + (67)(22)(1)_3! = \frac{1}{3}[(70)_3! - ((70 - 3)_3!(1)_3)!] = 2.571146697 \times 10^{33}.
\]

Theorem 5.4 If \( n \) is any positive integer, then

\[
\Delta_{k(\ell)}^{-1} \left[ k_\ell \left(\left[ \frac{k}{\ell} \right] + 1\right) \sum_{t=1}^{n} nC_t k^{n-t} \ell^t \right] = (k - \ell)\left(\left[ \frac{k}{\ell} \right]\right) k^n. \tag{33}
\]

Proof. The proof follows by (24) and Definition 4.1.

Theorem 5.5 If \( n \) is any positive integer, then

\[
\left[ \frac{k}{\ell} \right] \sum_{r=1}^{\left\lceil \frac{k}{\ell} \right\rceil} (k - \ell)^{(r-1)} ((k - r\ell)_\ell)! \left[ \sum_{t=1}^{n} nC_t k^{n-t} \ell^t \right] = (k - \ell)\left(\left[ \frac{k}{\ell} \right]\right) k^n. \tag{34}
\]

Proof. The proof follows from (33) and Definition 4.1.

Example 5.6 In (34), by taking \( n = 2 \), we find that

\[
\left[ \frac{k}{\ell} \right] \sum_{r=1}^{\left\lceil \frac{k}{\ell} \right\rceil} (k - \ell)^{(r-1)} ((k - r\ell)_\ell)\left[ \sum_{t=1}^{2} nC_t k^{n-t} \ell^t \right] = (k - \ell)\left(\left[ \frac{k}{\ell} \right]\right) k^2. \]

When, \( k = 61, \ell = 3 \) and \( j = 1 \), we get

\[
(58)_3^{(20)}(58)_3^{(0)}(357) + (55)_3^{(19)}(58)_3^{(1)}(339) + \cdots + (1)_3^{(1)}(58)_3^{(19)}(15) = (61 - 3)_3^{(20)}61^2 - (61 - 3)_3^{(20)}1^2 = 1.589854382 \times 10^{30}.
\]

Theorem 5.7 Let \( J = k - \left[ \frac{k}{\ell} \right] \ell \). Then, for all \( k \in \mathbb{N}_L(J) \),

\[
\left[ \frac{k}{\ell} \right] \sum_{r=1}^{\left\lceil \frac{k}{\ell} \right\rceil} (k - \ell)^{(r-1)} ((k - r\ell)_\ell)^{\left(\left[ \frac{k}{\ell} \right]-r\right)} = \frac{k_\ell\left(\left[ \frac{k}{\ell} \right]\right)}{\ell} - (k - \ell)\left(\left[ \frac{k}{\ell} \right]\right) \frac{J_\ell\left(\left[ \frac{k}{\ell} \right]\right)}{\ell}. \tag{35}
\]
Proof. From the equation (5) and Definition 4.1, we have

$$\Delta_{k\ell}^{-1}k_{\ell}^{\lceil \frac{j}{\ell} \rceil} = \frac{k_{\ell}^{\lceil \frac{j}{\ell} \rceil}}{\ell}. \quad (36)$$

The proof follows from (26) and (36).

Example 5.8 In (35), substituting $k = 62$, $\ell = 3$, $j = 2$ and $J = -1$, we get

$$\begin{align*}
(59)_{3}^{(0)} &+ (59)_{3}^{(1)}(56)_{3}^{(19)} + (59)_{3}^{(2)}(53)_{3}^{(18)} + \ldots + (59)_{3}^{(20)}(-1)_{3}^{(0)} \\
&= \frac{1}{3}[(62)_{3}^{(21)} - (59)_{3}^{(20)}(-1)_{3}^{(1)}] = 4.819811178X10^{28}.
\end{align*}$$

6 Conclusion

By selecting large value for $k$ and small value for $\ell > 0$, one can find the sum of several series easily using the Theorem mentioned above.

References


Received: December 15, 2014; Published: March 23, 2015