Graceful and Magic Labelings on Cayley Digraphs

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Abstract

A digraph $G(p,q)$ is edge odd graceful if there is a bijection $f : E(G) \rightarrow \{1,3,5,\ldots,2q-1\}$ such that when each vertex is assigned the sum of the labels of its outgoing arcs mod $2q$, the resulting vertex labels are distinct. A digraph $G(p,q)$ is said to be line graceful if the outgoing arcs of every vertex are labeled $0,1,2,\ldots,p-1$ then the resulting vertex label is the sum of the labels of the outgoing arcs of the vertex modulo $p$ and is distinct for every vertex. Let $k \geq 2$. A digraph $G(p,q)$ is called $\text{Mod}(k)$-edge magic if there is an edge labeling $f : E \rightarrow \{1,2,\ldots,q\}$ such that for each vertex $v$ the sum of the labels of the outgoing arcs of $v$ are equal to the same constant modulo $k$. In this paper we show that the Cayley digraphs are edge odd graceful, line graceful and mod $(k)$-edge magic.

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1 Introduction

Graph labeling was introduced by Rosa [4]. Bloom and Hsu [1,2] extended the graceful labeling concept on digraphs and established some relations between graceful digraphs and Latin squares, neo fields, Abelian groups and Galois fields. Lo [10] introduced the new methodology of labeling concept called edge graceful labeling. Lee [8] extended it as $k$-edge graceful labeling. In 2009 Solairaju and Chitra [12] introduced edge odd graceful labeling and discussed the same for Paths, Odd cycles and also for ladders. Gnanajothi [5] has defined a labeling parallel to edge graceful labeling and named line graceful labeling and she has in-
vestigated the line gracefulness for many undirected graphs. Magic labelings were introduced by Sedlacek [11]. k magic labeling is introduced by Lee [6, 9] and in the same paper they discussed about the integer magic spectrum of the k magic graph. The Mod (k) - edge-magic labeling was introduced by Lee, Seah and Tan in [7] and in the same paper they discussed that when k = p, the graph is Mod(p) – edge magic graph. The Cayley digraph that visualizes a group in the form of graph is introduced by Cayley [3]. Thirusangu, Atulya and Rajeswari [13] introduced total vertex magic labeling on some classes of Cayley digraphs.

The most wanted graphs in interconnection networks are vertex symmetric graphs. The Cayley digraphs are the vertex symmetric graphs which would be very useful in constructing models for interconnection networks. In this paper we discussed edge odd graceful labeling, line graceful labeling, mod (k) – edge magic and k - magic labeling for the Cayley digraphs.

2 Main Results

2.1. Graceful labelings on Cayley digraphs. In this section we define the edge odd graceful labeling and line graceful labeling for digraphs and also we discuss the existence of the above two labeling for Cayley digraphs and their line digraphs.

Definition 2.1.1. A digraph G(p,q) is edge odd graceful if there is a bijection f : E(G) → {1,3,5,…,2q-1} such that when each vertex is assigned the sum of the labels of its outgoing arcs under mod 2q, the resulting vertex labels are distinct.

Theorem 2.1.2. The Cayley digraph Cay (G,S) admits edge odd graceful labeling.

Proof: Consider the Cayley digraph with p vertices and m number of generators. Then the digraph Cay (G,S) has m outgoing and m incoming arcs at each vertex and totally the digraph has mp outgoing arcs.

Let us denote the vertex set of Cay(G,S) as V = {v1,v2,…,vp} and the edge set of Cay (G,S) as E = Es1 U Es2 U …… U Es_m = {e_{11},e_{12},…,e_{1m},e_{21},e_{22},…,e_{2m},…,e_{p1},…,e_{pm}}

Where e_{ij} is an outgoing arc from i^th vertex generated by the generators s_j and

E_{s_1} = set of all outgoing arcs from v_i generated by s_1
E_{s_2} = set of all outgoing arcs from v_i generated by s_2
.
.
.
E_{s_m} = set of all outgoing arcs from v_i generated by s_m

To prove that the Cayley digraph is edge odd graceful, define f: E(G) → {1,3,5,…,2q-1} as follows

f(e_{ij}) = 2(j-1)p + 2i - 1; 1≤ i ≤ p and 1≤ j ≤ m
Then for each vertex \( v_i, 1 \leq i \leq p \), the induced function

\[ f^*(v_i) = \sum_{j=1}^{m} f(e_{ij}) \]

\[ = 2i - 1 + 2p + 2i - 1 + 4p + 2i - 1 + \cdots + 2(m - 1)p + 2i - 1 \]

\[ f^*(v_i) \equiv \begin{cases}  
(2mi - m)(\text{mod } 2q) & \text{when } m \text{ is even} \\
(2mi - m + pq - q)(\text{mod } 2q) & \text{when } m \text{ is odd} 
\end{cases} \]

In both the cases \( f^* (v_i) \) is distinct for every \( v_i, 1 \leq i \leq p \).

Hence the Cayley digraph is edge odd graceful.

**Corollary 2.1.3.** The line digraph of Cay \((G, S)\) is also edge odd graceful.

**Proof:**

Consider the Cayley digraph Cay \((G, S)\) of \( m \) generators with \( p \) vertices and \( q \) arcs. Then by the definition of the line digraph, Line digraph of Cayley digraph is again an \( m \) regular digraph. That is every vertex of the line digraph has \( m \) incoming and \( m \) outgoing arcs. If the Cayley digraph of a group contains \( p \) vertices and \( q \) edges, then the corresponding line digraph contains \( q \) vertices and \( mq \) arcs. Let us denote the vertex set of \( L(\text{Cay}(G, S)) \) as

\[ V(L(G)) = \{v_1, v_2, v_3, \ldots, v_q\} \]

and denote the edge set of \( L(\text{Cay}(G, S)) \) as

\[ E(L(G)) = \{e_{11}, e_{12}, \ldots, e_{1m}, e_{21}, e_{22}, \ldots, e_{2m}, \ldots, e_{q1}, \ldots, e_{qm}\} \]

Define \( f: E(L(G)) \to \{1, 3, 5, \ldots, 2mq - 1\} \) as follows

\[ f(e_{ij}) = 2(j - 1)q + 2i - 1; \quad 1 \leq i \leq q \text{ and } 1 \leq j \leq m \]

Then for each vertex \( v_i, 1 \leq i \leq q \), the induced function \( f^*(v_i) = \sum_{j=1}^{m} f(e_{ij}) \)

\[ = 2mi - m + m(m - 1)q \]

This gives distinct values for every \( i \) under mod \( 2mq \).

Hence the line digraph of the Cayley digraph is also edge odd graceful.

**Definition 2.1.4.** A digraph \( G (p,q) \) with \( p \) vertices is said to be line graceful if there exists an injection \( f: E(G) \to \{0, 1, 2, \ldots, p\} \) such that the induced function \( f^* : V(G) \to \{0, 1, 2, \ldots, p - 1\} \) which is defined as \( f^*(v) = \sum_{v, e \in E(G)} f(e)(\text{mod } p) \) is bijective.

**Theorem 2.1.5** The Cayley digraph Cay \((G, S)\) with \( p \) vertices and \( |S| = m \) is line graceful provided, the g.c.d of \((m,p)\) = 1.
Proof:
From the construction of the Cayley digraph, we have $p$ vertices and $mp$ arcs where $m$ is the number of generators. Let us denote the vertex set of $\text{Cay}(G, S)$ as $V = \{v_1, v_2, \ldots, v_p\}$ and the edge set of $\text{Cay}(G, S)$ as

$$E = E_{s_1} \cup E_{s_2} \cup \ldots \cup E_{s_m} = \{e_{11}, e_{12}, \ldots, e_{1m}, e_{21}, e_{22}, \ldots, e_{2m}, \ldots, e_{p1}, \ldots, e_{pm}\}$$

Where $e_{ij}$ is an outgoing arc from the $i$th vertex generated by the generator $s_j$ and $E_{s_1} =$ set of all outgoing arcs from $v_i$ generated by $s_1$
$E_{s_2} =$ set of all outgoing arcs from $v_i$ generated by $s_2$
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$E_{s_m} =$ set of all outgoing arcs from $v_i$ generated by $s_m$

To prove that the Cayley digraph is line graceful, define $f: E(G) \rightarrow \{0, 1, 2, \ldots, p-1\}$ as follows

$$f(e_{ij}) = i + j - 2 \pmod{p}, 1 \leq i \leq p \text{ and } 1 \leq j \leq m$$

Then for each vertex $v_i$, $1 \leq i \leq p$, the induced function $f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$

$$= mi - 2m + 1 + 2 + \cdots + m$$

$$= \frac{m}{2} (2i - 3 + m)$$

Which gives distinct values for every $i$ under modulo $p$, if g.c.d of $(m, p) = 1$. Hence the Cayley digraph $\text{Cay}(G, S)$ is line graceful provided g.c.d of $(p, m) = 1$.

Corollary 2.1.6. If the Cay $(G, S)$ is line graceful, then the line digraph of Cay $(G, S)$ is not line graceful.

Proof:
Consider the Cayley digraph of $m$ generators $\text{Cay}(G, S)$ with $p$ vertices and $mp$ arcs. Then by the definition of the line digraph, Line digraph of Cayley digraph is again a $m$ regular digraph. That is every vertex of the line digraph has $m$ incoming and $m$ outgoing arcs. If the Cayley digraph of a group contains $p$ vertices and $mp$ edges, then the corresponding line digraph has $mp$ vertices and $m^2p$ arcs. Therefore the g.c.d of $(mn, m) \neq 1$.

Hence the Cay $(G, S)$ is line graceful, then the line digraph of Cay $(G, S)$ is not line graceful.

2.2. Magic labelings on Cayley digraphs. In this section we discuss the $k$ magic and mod $(k)$ - edge magic labelings on Cayley digraphs.
**Definition 2.2.1.** A digraph is said to be $k$ magic if there is a labeling from the edges of $G$ to the set \{1,2,\ldots,k-1\} such that for each vertex $v$ of $G$, the sum of the labels of the outgoing arcs is constant independent of $v$. The set of all $k$ for which $G$ is $k$ magic is called integer magic spectrum of $G$ and is denoted by $\text{IM}(G)$.

The Cayley digraph $\text{Cay}(G,S)$ is the digraph with $p$ vertices and $m$ number of generators. i.e., Every vertex of the Cayley digraph has $m$ outgoing and $m$ incoming arcs. Since it is a regular directed graph, it admits $k$ magic labeling for $k = 1, 2, 3\ldots$ Therefore the integer spectrum of Cayley digraph is the set of all natural numbers.

$$\text{IM}(\text{Cay}(G,S)) = \mathbb{N}\setminus\{1,2,\ldots,m-1\}$$

**Definition 2.2.2.** Let $k \geq 2$. A $(p,q)$ digraph $G$ is called Mod$(k)$ – edge magic if there is an edge labeling $f: E \rightarrow \{1,2,\ldots,q\}$ such that for each vertex $v$ the sum of the labels of the outgoing arcs of $v$ is equal to the same constant modulo $k$.

**Lemma 2.2.3.** Every regular directed magic graph is mod$(k)$ – edge magic for any $k \geq 2$.

**Proof:**

We know that if the $m$ regular directed graph $G(p,q)$ is magic, then $\sum_{j=1}^{m} f(e_{ij}) = r$, a constant for $1 \leq i \leq p$ where $f(e_{ij})$ is the label of the $j^{\text{th}}$ outgoing arc of $i^{\text{th}}$ vertex. For $k \geq 2$

$$\sum_{j=1}^{m} f(e_{ij}) = r \equiv c \pmod{k} \text{ for } 1 \leq i \leq p \text{ where } c \in Z_k.$$  

Hence the regular directed magic graph is mod$(k)$ – edge magic for any $k \geq 2$.

**Theorem 2.2.4.** The Cayley $(G, S)$ is mod$(k)$ – edge magic, if $|S| \equiv 0 \pmod{2}$.

**Proof:**

We know that the Cayley digraph with even number of generators is magic. Since the Cayley digraph is regular and magic, by lemma 2.2.3 we can say that the $\text{Cay}(G,S)$ is mod$(k)$ – edge magic if $|S| \equiv 0 \pmod{2}$.

**Theorem 2.2.5.** The Cayley digraph $\text{Cay}(G, S)$ with $|S| \equiv 1 \pmod{2}$ admits mod$(k)$ – edge magic labeling, if $k = 3,5,\ldots,|S|$.

**Proof:**

Consider the Cayley digraph with $p$ vertices and $q$ arcs. Let $|S| = m$, then we have every vertex of $\text{Cay}(G, S)$ has $m$ number of outgoing and $m$ number of incoming arcs where $m$ is odd and $q=mp$. Let us denote the vertex set of $\text{Cay}(G,S)$ as $V = \{v_1, v_2,\ldots,v_p\}$ and the edge set of $\text{Cay}(G,S)$ as $E = E_{s_1} \cup E_{s_2} \cup \ldots \cup E_{s_m}$

$$= \{e_{11}, e_{12},\ldots,e_{1m},e_{21},e_{22},\ldots,e_{2m},\ldots,e_{p1},\ldots,e_{pm}\}.$$  

Where $e_{ij}$ is outgoing arc of $i^{\text{th}}$ vertex generated by $s_j$. Now we have to show that the Cayley digraph $\text{Cay}(G,S)$ with $|S| \equiv 1 \pmod{2}$ admits mod$(k)$ – edge magic labeling, if $k = 3,5,\ldots,m$, i.e., we have to show that if $f: E \rightarrow \{1,2,\ldots,q\}$ then $\sum_{j=1}^{m} f(e_{ij}) \equiv c \pmod{k}$ for $k = m, m-2, m-4,\ldots,5,3$ and for $1 \leq i \leq p$. 

We prove this theorem in two cases for \( k = m \) and for \( k = m - d \), where \( d = 2, 4, 6, \ldots, m-3 \).

**Case (i): For \( k = m \):**
Define \( f: E \to \{1,2,\ldots,q\} \) as follows
\[
f(e_{ij}) = (j - 1)p + i; \quad \text{for } 1 \leq i \leq p \text{ and } 1 \leq j \leq m
\]
Then for each vertex \( v_i, 1 \leq i \leq p \)
\[
\sum_{j=1}^{m} f(e_{ij}) = mi + p(1 + 2 + 3 + \cdots + m - 1) = mi + \frac{p(m-1)m}{2} \quad \text{for } 1 \leq i \leq p
\]

Since \( m \) is odd and \((m-1)/2\) is an integer, we have \( \sum_{i=1}^{m} f(e_{ij}) \equiv 0 \pmod{m} \) for all \( i, 1 \leq i \leq p \). Hence the Cayley digraph \( \text{Cay} \ (G,S) \) with \( |S| \equiv 1 \pmod{2} \) admits \( \text{mod}(k) \) – edge magic labeling, if \( k = m \).

**Case (ii): For \( k = m - d \), where \( d = 2, 4, 6, \ldots, m-3 \) :**
Define \( f: E \to \{1,2,\ldots,q\} \) as follows
\[
f(e_{ij}) = \begin{cases} 
(j - 1)p + i & \text{for } 1 \leq j \leq m \text{ and } j \neq 2, 4, \ldots, d \\
p + 1 - i & \text{for } 1 \leq j \leq m \text{ and } j = 2, 4, \ldots, d 
\end{cases}
\]
Then for each vertex \( v_i, 1 \leq i \leq p \)
\[
\sum_{j=1}^{m} f(e_{ij}) = \sum_{j=1}^{m} f(e_{ij}) + \sum_{j=2}^{d} f(e_{ij}) + \sum_{j=d+2}^{m-1} f(e_{ij})
\]
\[
= \left[ 2p + 4p + \cdots + (m - 1)p \right] + \frac{m + 1}{2} (i) + \left[ 2p + 4p + \cdots + dp \right] + \frac{d}{2} - \frac{d}{2} (i)
\]
\[
+ [(d + 1)p + (d + 3)p + \cdots + (m - 2)p] + \left( m - \frac{d + m + 1}{2} \right) (i)
\]
\[
= i \left[ m + 1 - d + 2m - d - m - 1 \right] + \frac{d}{2} + \frac{(2p)(m-1)(m+1)}{4 \times 2} - \frac{p(m-1-d)}{2}
\]
\[
= i(m - d) + \frac{d}{2} + \frac{p(m-1)(m+1)}{4} + \frac{2p(m-1)(m+1)}{4 \times 2} - \frac{p(m-1-d)}{2}
\]
\[
= i(m - d) + \frac{d}{2} + \frac{p(m-d)}{2} + \frac{p(m-1)m}{2} + \frac{p(m-1)}{2}
\]
\[= i(m - d) + \frac{d}{2} + \frac{p}{2} + \frac{pm}{2} - \frac{p}{2} - \frac{p(m - d)}{2} + \frac{p(m - 1)}{2}(m - d + d)\]

\[= i(m - d) + \frac{p}{2} (m - 1)(m - d) + \frac{dp}{2} (m - d + d)\]

\[= i(m - d) + \frac{p}{2} (m - 1)(m - d) + \frac{dp}{2} (m - d) + \frac{d}{2}(1 + dp)\]

Since \(d\) is even and \((m-1)/2\) is an integer, we have

\[
\sum_{j=1}^{m} f(e_{ij}) \equiv \frac{d}{2} (1 + dp) \pmod{(m-d)} \text{ for all } 1 \leq i \leq p.
\]

Hence the Cayley digraph \(\text{Cay}(G, S)\) with \(|S| \equiv 1 \pmod{2}\) admits \(\text{mod}(k)\) – edge magic labeling, if \(k = m-d\); where \(d = 2, 4, \ldots, m-3\).

Hence we proved that The Cayley digraph \(\text{Cay}(G,S)\) with \(|S| \equiv 1 \pmod{2}\) admits \(\text{mod}(k)\) – edge magic labeling, if \(k = 3, 5, \ldots, |S|\).

**Corollary 2.2.6.** From the above results we can say that every magic regular directed graph is \(\text{mod}(k)\) – edge magic. The converse is not true, since the Cayley digraph with odd number of generators is not magic.

### 3 Conclusion and Open Problems

In this paper we discussed edge odd graceful, line graceful, k-magic and \(\text{mod}(k)\) – edge magic labeling on Cayley digraphs. We are currently investigating new labeling methodologies on Cayley networks. In future we can also implement these labeling on other networks such as De Bruijn and Kautz networks.

### References


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