Separation Axioms by $\Delta^*$-Closed Sets
in Topological Spaces

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Abstract

In this paper the separation axioms by $\Delta^*$-closed sets namely $\Delta^{*T\delta}$-space, $\Delta^{*T\delta^*}$-space, $g\delta^{T\Delta^*}$-space and $\delta g^{#T\Delta^*}$-space are introduced and their properties are discussed.

Mathematics Subject Classification: 54C10, 54D10

Keywords: $\delta g^*$-closed sets, $g\delta$-closed sets, $\delta g^#$-closed sets and $\Delta^*$-closed sets

1 Introduction

Julian Dontchev [1] offered a class of generalized closed sets called \( \delta g \)-closed sets in 1996. R.Sudha and K.Sivakamasundari [10] established \( \delta g \)-\( * \)-closed sets in 2012. A new class of generalised closed sets called \( \Delta ^* \)-closed sets in topological spaces using \( \delta g \)-closed sets was introduced in 2014 by K.Meena and K.Sivakamasundari [7]. The objective of this paper is to contribute the separation axioms by \( \Delta ^* \)-closed sets. Also we study their basic properties. Throughout this paper \((X, \tau)\) represents a non empty topological space on which no separation axioms are mentioned unless otherwise specified.

In the previous paper [7] of K.Meena and K.Sivakamasundari, \( \Delta ^* \)-closed sets were denoted by \( \delta (\delta g)^* \)-closed sets.

## 2 Preliminaries

**Definition 2.1** A subset \( A \) of a topological space \((X, \tau)\) is called \( \Delta ^* \)-closed set [7] if \( \delta cl (A) \subseteq U \) whenever \( A \subseteq U \), \( U \) is \( \delta g \)-open in \((X, \tau)\). The class of all \( \Delta ^* \)-closed sets of \((X, \tau)\) is denoted by \( \Delta ^* C(X, \tau) \).

**Definition 2.2** The closure operator of \( \Delta ^* \)-closed set is defined as \( \Delta ^* cl (A) = \cap \{ F \subseteq X : A \subseteq F \text{ and } F \text{ is } \Delta ^* \text{-closed in } (X, \tau) \}\).[8]

**Definition 2.3** For any subset \( U \) of \((X, \tau)\), a new class of sets denoted by \( \Delta ^* \tau ^# \) is defined using \( \Delta ^* \) closure operator as \( \Delta ^* \tau ^# = \{ U : \Delta ^* cl (X - U) = X - U \} \).[8]

**Definition 2.4** A topological space \((X, \tau)\) is said to be a

1) \( T_{1/2} \)-space if every \( g \)-closed subset of \((X, \tau)\) is closed in \((X, \tau)\).[6]
2) \( T_{3/4} \)-space if every \( \delta g \)-closed subset of \((X, \tau)\) is \( \delta \)-closed in \((X, \tau)\).[1]
3) \( T_3 \)-space if every \( g s \)-closed subset of \((X, \tau)\) is closed in \((X, \tau)\).[3]
4) \( T_c \)-space if every \( g s \)-closed subset of \((X, \tau)\) is \( g ^* \)-closed in \((X, \tau)\).[13]
5) \( T_g \)-space if every \( g s \)-closed subset of \((X, \tau)\) is \( g \)-closed in \((X, \tau)\).[3]
6) \( \alpha T_b \)-space if every \( \alpha g \)-closed subset of \((X, \tau)\) is closed in \((X, \tau)\).[4]
7) \( \alpha T_c \)-space if every \( \alpha g \)-closed subset of \((X, \tau)\) is \( g ^* \)-closed in \((X, \tau)\).[13]
8) \( * T_{1/2} \)-space if every \( g \)-closed subset of \((X, \tau)\) is \( g ^* \)-closed in \((X, \tau)\).[13]
9) \( T_{\delta } \)-space if every \( g \delta \)-closed subset of \((X, \tau)\) is \( \delta \)-closed in \((X, \tau)\).[2]
10) \( \delta g ^{T_3} \)-space if every \( \delta g ^* \)-closed subset of \((X, \tau)\) is \( \delta \)-closed in \((X, \tau)\).[11]
11) \( \delta g ^{T_{3/2} \delta ^*} \)-space if every \( \delta g \)-closed subset of \((X, \tau)\) is \( \delta g ^* \)-closed in \((X, \tau)\).[11]
12) \( g s ^{T_{3/2} \delta ^*} \)-space if every \( g s \)-closed subset of \((X, \tau)\) is \( \delta g ^* \)-closed in \((X, \tau)\).[11]
13) \( g \delta ^{T_{3/2} \delta ^*} \)-space if every \( g \delta \)-closed subset of \((X, \tau)\) is \( \delta g ^* \)-closed in \((X, \tau)\).[11]
14) \( \alpha g ^{T_{3/2} \delta ^*} \)-space if every \( \alpha g \)-closed subset of \((X, \tau)\) is \( \delta g ^* \)-closed in \((X, \tau)\).[11]
15) \( \delta g ^{T_{3/2} \delta ^*} \)-space if every \( \delta g \)-closed subset of \((X, \tau)\) is \( \delta g ^* \)-closed in \((X, \tau)\).[11]
16) \( \alpha \delta g ^{T_{3/2} \delta ^*} \)-space if every \( \alpha g \)-closed subset of \((X, \tau)\) is \( \delta g ^* \)-closed in \((X, \tau)\).[11]
3 Separation Axioms

Definition 3.1 A space \((X, \tau)\) is said to be a
1) \(\Delta^*{T_\delta}\)-space if every \(\Delta^*\)-closed subset of \((X, \tau)\) is \(\delta\)-closed in \((X, \tau)\).
2) \(\Delta^*{T_{\delta^*}}\)-space if every \(\Delta^*\)-closed subset of \((X, \tau)\) is \(\delta^*\)-closed in \((X, \tau)\).
3) \(g\delta{T_{\Delta^*}}\)-space if every \(g\delta\)-closed subset of \((X, \tau)\) is \(\Delta^*\)-closed in \((X, \tau)\).
4) \(\delta^*{g}\#T_{\Delta^*}\)-space if every \(\delta^*g\#\)-closed subset of \((X, \tau)\) is \(\Delta^*\)-closed in \((X, \tau)\).

\(\Delta^*{T_\delta}\)-space

Proposition 3.2 If \((X, \tau)\) is a \(\Delta^*{T_\delta}\)-space then \(\Delta^*cl(B) = \delta cl(B)\) for each subset \(B\) of \(X\).

Proof: Let \((X, \tau)\) be a \(\Delta^*{T_\delta}\)-space. We have already proved that every \(\delta\)-closed set is \(\Delta^*\)-closed set (By Proposition 3.2[8]). Moreover \((X, \tau)\) is a \(\Delta^*{T_\delta}\)-space. Therefore \(\Delta^*CL(X, \tau) = \delta CL(X, \tau)\). Hence by definition of \(\delta\)-closure and \(\Delta^*\)-closure, \(\Delta^*cl(B) = \delta cl(B)\) for each subset \(B\) of \(X\).

Theorem 3.3 The following statements are equivalent for a space \((X, \tau)\).

a) \((X, \tau)\) is \(\Delta^*{T_\delta}\)-space.

b) \(\tau_\delta = \Delta^*\tau^\#\)-holds.

c) Every singleton \(\{x\}\) is either \(\delta g\)-closed or \(\delta\)-open.

d) Every singleton \(\{x\}\) is either \(\delta g\)-closed or regular open.

Proof: (a)⇒(b): We claim that \(\Delta^*\tau^\# \subseteq \tau_\delta\). Let \(V \in \Delta^*\tau^\#\). By assumption and by proposition 3.2 \(\Delta^*cl(U) = \delta cl(U)\) for every subset \(U\) of \(X\). Therefore \(\Delta^*cl(X - V) = \delta cl(X - V) = X - V\) by definition of \(\Delta^*\tau^\#\). Hence \(V \in \tau_\delta\).

We know that for a subset \(A\) of \((X, \tau)\), \(A \subseteq \Delta^*cl(A) \subseteq \delta cl(A)\). So by the definition of \(\Delta^*cl(A)\) for any topology \(\tau\) we get \(\tau_\delta \subseteq \Delta^*\tau^\#\). Hence \(\tau_\delta = \Delta^*\tau^\#\).

(b)⇒(c): Let \(x \in X\). We know that by proposition 3.11 of [11] if \(\{x\}\) is not \(\delta g\)-closed then \(X - \{x\}\) is \(\Delta^*\)-closed. Therefore \(\Delta^*cl(X - \{x\}) = X - \{x\}\) which implies that \(\{x\} \in \Delta^*\tau^\# = \tau_\delta\) by (b). Therefore \(\{x\}\) is \(\delta\)-open.

(c)⇒(d): The proof follows from the fact that in any space a singleton is \(\delta\)-open if and only if it is regular open.

Proposition 3.4 If \((X, \tau)\) is a \(\Delta^*{T_\delta}\)-space then for every subset \(A\) of \(X\) \(\Delta^*cl(A)\) is \(\delta\)-closed in \((X, \tau)\).

Proof: Since \((X, \tau)\) is a \(\Delta^*{T_\delta}\)-space, by definition of \(\Delta^*cl(A)\), \(\Delta^*cl(A)\) is \(\delta\)-closed in \((X, \tau)\).
Proposition 3.5 Every $T_\delta$-space is a $\Delta^*T_\delta$-space but not conversely.

Proof: Let $A$ be $\Delta^*$-closed in $(X, \tau)$. Since every $\Delta^*$-closed set is $g\delta$-closed in $(X, \tau)$ and $(X, \tau)$ is a $T_\delta$-space, $A$ is $\delta$-closed. Hence $(X, \tau)$ is a $\Delta^*T_\delta$-space.

Counter example 3.6 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Then $(X, \tau)$ is a $\Delta^*T_\delta$-space but not $T_\delta$-space since the subset $\{b\}$ is $g\delta$-closed but not $\delta$-closed in $(X, \tau)$.

Remark 3.7 The spaces $T_c$ and $T_d$ are independent with $\Delta^*T_\delta$-space as seen from the following examples.

Example 3.8 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}\}$. Then $(X, \tau)$ is a $\Delta^*T_\delta$-space but not $T_c$-space and $T_d$-space since the subset $\{b\}$ is $gs$-closed but not $g^*$-closed and not $g$-closed in $(X, \tau)$.

Example 3.9 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then $(X, \tau)$ is a $T_c$-space and $T_d$-space but not $\Delta^*T_\delta$-space since the subset $\{c\}$ is $\Delta^*$-closed but not $\delta$-closed in $(X, \tau)$.

Remark 3.10 The $\Delta^*T_\delta$-space is independent with $\alpha T_b$-space as seen from the following examples.

Example 3.11 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Then $(X, \tau)$ is a $\alpha T_b$-space but not $\Delta^*T_\delta$-space since the subset $\{b, c\}$ is $\Delta^*$-closed but not $\delta$-closed in $(X, \tau)$.

Example 3.12 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Then $(X, \tau)$ is a $\Delta^*T_\delta$-space but not $\alpha T_b$-space since the subset $\{a, c\}$ is $\alpha g$-closed but not closed in $(X, \tau)$.

Similarly the following results are true.

Remark 3.13 i) The $\Delta^*T_\delta$-space is independent with $\alpha T_c$-space.
ii) The $\Delta^*T_\delta$-space is independent with $*T_{1/2}$-space.
iii) The space $\Delta^*T_\delta$-space is independent with $T_b$-space.

Proposition 3.14 If $(X, \tau)$ is a $\delta g T_{3g^*}$-space and $\Delta^*T_\delta$-space then it is a $T_{3/4}$-space.

Proof: Let $A$ be $\delta g$-closed in $(X, \tau)$. Since $(X, \tau)$ is $\delta g T_{3g^*}$-space, $A$ is $\delta g^*$-closed. Since every $\delta g^*$-closed is $\Delta^*$-closed and $(X, \tau)$ is $\Delta^*T_\delta$-space, $A$ is
$\delta$-closed. Hence $(X, \tau)$ is a $T_{3/4}$-space.

**Proposition 3.15** Every $g\delta T_{\delta^{g^*}}$-space is $\Delta^* T_{3}$-space but not conversely.

**Proof:** Let $X$ be a $T_{\delta}$-space. Every $T_{\delta}$-space is $\Delta^* T_{3}$-space (Proposition 3.5). Also we know that every $T_{\delta}$-space is $g\delta T_{\delta^{g^*}}$-space. Hence every $g\delta T_{\delta^{g^*}}$-space is $\Delta^* T_{3}$-space.

**Counter example 3.16** Let $X = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{b, c\}\}$. Then $(X, \tau)$ is a $\Delta^* T_{3}$-space. The subset $\{b\}$ is $g\delta$-closed but not $\delta g^*$-closed in $(X, \tau)$. Hence $(X, \tau)$ is not a $g\delta T_{\delta^{g^*}}$-space.

**$\Delta^* T_{\delta^{g^*}}$-space**

**Proposition 3.17** Every $\Delta^* T_{\delta}$-space is $\Delta^* T_{\delta^{g^*}}$-space but the converse is not true.

**Proof:** Let $A$ be $\Delta^*$-closed in $(X, \tau)$. Since $(X, \tau)$ is $\Delta^* T_{\delta}$-space $A$ is $\delta$-closed. Moreover every $\delta$-closed is $\delta g^*$-closed in $(X, \tau)$. Hence $(X, \tau)$ is a $\Delta^* T_{\delta^{g^*}}$-space.

**Counter example 3.18** Let $X = \{a, b, c\}, \tau = \{\phi, X, \{a, b\}\}$. Then $(X, \tau)$ is $\Delta^* T_{\delta^{g^*}}$-space but not $\Delta^* T_{\delta}$-space since the subset $\{c\}$ is $\Delta^*$-closed but not $\delta$-closed in $(X, \tau)$.

**Proposition 3.19** If $(X, \tau)$ is a $\Delta^* T_{\delta}$-space and a $g s T_{\delta^{g^*}}$-space then it is a $\Delta^* T_{\delta^{g^*}}$-space.

**Proof:** Let $A$ be $\Delta^*$-closed in $(X, \tau)$. Then the proof of i)-v) follows from the fact that every $\delta$-closed set is $\delta g^*$-closed, $\alpha g^*$-closed, $g\delta$-closed, $\alpha g$-closed and $\delta g$-closed in $(X, \tau)$.
Proposition 3.21 If \((X, \tau)\) is a \(\Delta^*T_{s_0^*}\)-space and \(T_{3/4}\)-space then it is a \(\delta g^*T_{\delta}\)-space.

Proof: Let \(A\) be \(\Delta^*\)-closed in \((X, \tau)\). Since \((X, \tau)\) is a \(\Delta^*T_{s_0^*}\)-space, \(A\) is \(\delta g^*\)-closed. We know that every \(\delta g^*\)-closed is \(\delta g\)-closed. Therefore \(A\) is \(\delta g\)-closed. Also \((X, \tau)\) is a \(T_{3/4}\)-space. So \(A\) is \(\delta\)-closed in \((X, \tau)\). Hence \((X, \tau)\) is a \(\delta g^*T_{\delta}\)-space.

Remark 3.22 The space \(\Delta^*T_{s_0^*}\)-space is independent with the spaces \(T_b, T_c, T_d\)-spaces as seen from the following examples.

Example 3.23 Let \(X = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{a, b\}\}\). Then \((X, \tau)\) is a \(\Delta^*T_{s_0^*}\)-space but not \(T_b, T_c, T_d\)-spaces since the subset \(\{b\}\) is \(gs\)-closed but not \(\delta g\)-closed, \(g\)-closed and \(g\)-closed sets respectively for \(T_b, T_c, T_d\)-spaces.

Example 3.24 Let \(X = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}\). Then \((X, \tau)\) is a \(T_b, T_c, T_d\)-spaces but not \(\Delta^*T_{s_0^*}\)-space since the subset \(\{b\}\) is \(\Delta^*\)-closed but not \(\delta g^*\)-closed set in \((X, \tau)\).

\(g\delta^T\Delta^*\)-space

Proposition 3.25 If \((X, \tau)\) is a \(g\delta^T\Delta^*\)-space and a \(\Delta^*T_{\delta}\)-space then it is a \(T_{\delta}\)-space.

Proof: Let \(A\) be \(g\delta\)-closed in \((X, \tau)\). Since \((X, \tau)\) is a \(g\delta^T\Delta^*\)-space \(A\) is \(\Delta^*\)-closed in \((X, \tau)\). Also \((X, \tau)\) is a \(\Delta^*T_{\delta}\)-space. Therefore \(A\) is \(\delta\)-closed in \((X, \tau)\). Hence \((X, \tau)\) is a \(T_{\delta}\)-space.

Proposition 3.26 If \((X, \tau)\) is a \(g\delta^T\Delta^*\)-space and a \(\Delta^*T_{s_0^*}\)-space then it is a \(g\delta T_{s_0^*}\)-space.

Proof: Let \(A\) be \(g\delta\)-closed in \((X, \tau)\). Since \((X, \tau)\) is a \(g\delta^T\Delta^*\)-space \(A\) is \(\Delta^*\)-closed in \((X, \tau)\). Also \((X, \tau)\) is a \(\Delta^*T_{s_0^*}\)-space. Therefore \(A\) is \(\delta g^*\)-closed in \((X, \tau)\). Hence \((X, \tau)\) is a \(g\delta T_{s_0^*}\)-space.

\(\delta g^\#T\Delta^*\)-space

Proposition 3.27 If \((X, \tau)\) is a \(T_{\delta}\)-space and a \(\delta g^\#T\Delta^*\)-space then it is a \(g\delta^T\Delta^*\)-space.
**Proof:** Let $A$ be $g\delta$-closed in $(X, \tau)$. Since $(X, \tau)$ is a $T_\delta$-space, $A$ is $\delta$-closed and hence it is $\delta g^\#$-closed. Also $(X, \tau)$ is a $\delta g^\# T_{\Delta^*}$-space. Therefore $A$ is $\Delta^*$-closed and hence $(X, \tau)$ is a $g\delta T_{\Delta^*}$-space.

**Remark 3.28** The spaces $T_b, T_c$ and $T_d$ are independent with $\delta g^\# T_{\Delta^*}$-space as seen from the following examples.

**Example 3.29** Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then $(X, \tau)$ is a $\delta g^\# T_{\Delta^*}$-space but not $T_b, T_c$ and $T_d$-spaces. Since the subset $\{a\}$ is $gs$-closed but not closed, $g^*$-closed and $g$-closed sets in $(X, \tau)$ for $T_b, T_c$ and $T_d$-spaces respectively.

**Example 3.30** Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then $(X, \tau)$ is a $T_b, T_c$ and $T_d$-space but not $\delta g^\# T_{\Delta^*}$-space since the subset $\{a, b\}$ is $\delta g^\#$-closed but not $\Delta^*$-closed in $(X, \tau)$.

**References**


Received: December 15, 2014; Published: March 23, 2015