On Vertex Coloring of Graphs

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Abstract

The concept of vertex coloring pose a number of challenging open problems in graph theory. Among several interesting parameters, the coloring parameter, namely the pseudoachromatic number of a graph stands a class apart. Although not studied very widely like other parameters in the graph coloring literature, it has started gaining prominence in recent years. The pseudoachromatic number of a simple graph \( G \), denoted \( \psi(G) \), is the maximum number of colors used in a vertex coloring of \( G \), where the adjacent vertices may or may not receive the same color but any two distinct pair of colors are represented by at least one edge in it. In this paper we have computed this parameter for a number of classes of graphs.

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1 Introduction

In this paper, we are concerned with finite, simple and undirected graphs.
A main interest in graph theory is to probe the nature of action of any parameter in graphs. Coloring of graphs are very extended areas of research. A coloring of a graph can be described by a function that maps elements of a graph (vertices-vertex coloring, edges-edge coloring or both-total coloring) into some set of numbers (possibly $N$, $Z$ or even $R$) usually called colors such that some property is satisfied.

A proper or chromatic vertex coloring of a graph $G$ with vertex set $V$ and edge set $E$ is a map $f : V(G) \to S$ such that $f(u) \neq f(v)$ whenever $uv \in E(G)$. The smallest number of colors used in such a coloring is called the chromatic number of $G$, denoted by $\chi(G)$.

By a pseudocoloring we mean a coloring of the elements of a vertex set $V(G)$ of $G$ in which any two adjacent vertices may or may not receive the same color. Observe that in such a pseudocoloring, a subgraph induced by an arbitrary color class need not be a null graph. A pseudoachromatic coloring of a graph $G$ is a pseudocoloring of the vertices of $G$, such that, for any pair of distinct colors, there is at least one edge whose end vertices are colored with this pair of colors. The pseudoachromatic number denoted by $\psi$ of a graph $G$ is the maximum number of colors involved in a pseudoachromatic coloring of $G$. A pseudoachromatic coloring of $G$ is called optimal if in such a coloring only $\psi(G)$ colors are used.

A complete coloring of a graph $G$ is a proper vertex coloring of $G$ which is also pseudoachromatic. The greatest number of colors used in a complete coloring of $G$ is the achromatic number $\alpha(G)$, of $G$. It is clear from the definition that $\chi(G) \leq \alpha(G) \leq \psi(G)$.

Harary et. al in [12] coined the term achromatic number. The pseudoachromatic number was due to Gupta [11]. Several researchers in the past have dealt with these parameters and it is quite difficult to compute then exactly. For more, one can see [3, 6] and the references therein. The computation of achromatic and pseudoachromatic indices of complete graphs for certain special cases were dealt with in [3, 6]. Aichholzer et. al in [1] have extended the idea of computation of these indices to complete geometric graphs.

By a cartesian product of two graphs $G, H$, denoted $G \times H$, we mean a graph with vertex set $V(G) \times V(H)$ where any two arbitrary vertices $(u, v)$ and $(u', v')$ are said to be adjacent to each other if and only if either $u = u'$ and $(v, v') \in E(H)$ or $(u, u') \in E(G)$ and $v = v'$.

Given a graph $G$, form a new graph, denoted $\text{cor}(G)$ and called corona of $G$ by introducing for each vertex $u$ in $G$ a new vertex that is adjacent only to $u$. The generalized Petersen graph $P(n, k)$ for $n \geq 5$, $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ has vertices, and respectively, edges given by $V(P(n, k)) = \{a_i, b_i : 0 \leq i \leq n-1\}$, $E(P(n, k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} : 0 \leq i \leq n-1\}$, where the subscripts are expressed as integers modulo $n$ ($n \geq 5$).

Let $G$ be a graph of order $n$ and $\alpha$ be a permutation on the set $\{1, \ldots, n\}$. 
The permutation graph $G^*$ of the graph $G$ is the graph that consists of two disjoint copies of $G$, say, $G_1, G_2$ with $n$ more edges $e_{i, \alpha(i)}$ that join the vertex $u_i \in G_1$ with the vertex $v_{\alpha(i)} \in G_2$.

2 Main Results

**Theorem 2.1.** [16] The pseudoachromatic number of $K_{m,n}$ is $\min(m,n)+1$.

**Corollary 2.2.** The pseudoachromatic number of $K_{1,n}$ is 2.

**Theorem 2.3.** The pseudoachromatic number of the permutation graph of $K_{1,n}$ with respect to the identity permutation $I$ is $\psi + 1$, where $\psi = \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$ if $n = \psi(\psi - 1)$.

**Theorem 2.4.** The pseudoachromatic number of the permutation graph of $K_{1,n}$ with respect to any permutation $\alpha$ is $\psi + 1$ if $n = \psi(\psi - 1)$.

**Theorem 2.5.** Let $G$ be the permutation graph of $K_{m,n}$ with $n \geq 2m$ under some permutation $\alpha$ on the set of vertices of one copy of $K_{m,n}$. Then

$$
\psi(G) = \begin{cases} 
  n + 1 & \text{if no permutation allots a vertex with color } (n + 1) \text{ and to a vertex with color } (n + 2) \\
  n + 2 & \text{if there exists a permutation that allots a vertex with color } (n + 1) \text{ to a vertex with color } (n + 2).
\end{cases}
$$

**Theorem 2.6.** Let $G$ be any graph. Then the 1-crown graph $G \odot K_{1,1}$ obtained from $G$ by identifying the central vertex of $K_{1,1}$ with each vertex of $G$ has the pseudoachromatic number $\psi + 1$.

**Theorem 2.7.** The bi-star graph $B_{n,n}$ obtained from two disjoint copies of $K_{1,n}$ by joining their centres by an edge has pseudoachromatic number 3.

**Corollary 2.8.** The bi-star $B_{m,n}$ obtained from $K_{1,m}$ and $K_{1,n}$ by joining their centres by an edge has pseudoachromatic number 3.

**Theorem 2.9.** [15] $\psi(P_n) = 2k - 1$, $2k$ or $2k + 1$ according as $k(2k - 1) \leq p \leq k(2k - 1) + (k - 1)$ or $k(2k - 1) + k \leq p \leq k(2k + 1)$ or $k(2k + 1) + 1 \leq p \leq (k + 1)(2k + 1)$.

**Corollary 2.10.** The umbrella graph $U_n$ obtained from disjoint copies of $K_2$ and $P_n$ by joining each vertex of $P_n$ with any one vertex of $K_2$ has pseudoachromatic number $\psi(P_n) + 1$. 
Theorem 2.11. \[15\] \( \psi(C_p) = 2k - 1, 2k \text{ or } 2k + 1 \) according as \( k(2k - 1) \leq p \leq k(2k - 1) + (k - 1) \text{ or } k(2k - 1) + k \leq p \leq k(2k + 1) - 1 \text{ or } k(2k + 1) \leq p \leq (k + 1)(2k + 1) - 1 \).

Corollary 2.12. The Hoffman tree \( P_n \odot K_1 \), a graph obtained from the path \( P_n \) by attaching a pendant edge at each vertex of \( P_n \) has pseudoachromatic number \( \psi(P_n) + 1 \).

Corollary 2.13. \( \psi(C_p \times K_2) \geq \psi(C_{2p}) \).

Proposition 2.14. The Jelly fish graph \( J(m, n) \) has pseudoachromatic number 4.

Theorem 2.15. \( \psi(\text{cor}(G) \times \text{cor}(H)) \geq \psi(C_{2(p_1 + p_2)}) \) provided \( G \times H \) is hamiltonian of even order for any two graphs \( G \) and \( H \) with \( |V(G)| = p_1 \) and \( |V(H)| = p_2 \).

Theorem 2.16. If \( G \) is a self complementary graph of order \( p \) then \( 2\psi(G) \leq \begin{cases} 1 + \sqrt{1 + 32k} & \text{if } p = 4k \\ 1 + \sqrt{32k + 9} & \text{if } p = 4k + 1. \end{cases} \)

Define \( s_m = 2r^2 \) if \( m = 2r \) and \( 2r^2 - 3r + 1 \) if \( m = 2r - 1 \). Then it is easy to note that \( s_m \) is a monotonically increasing sequence of non negative integers.

Theorem 2.17. \([5]\) Consider the cycle graph on \( p \geq 3 \) vertices, denoted by \( C_p \). Find the positive integer \( m \) such that \( s_m \leq p \leq s_{m+1} \). If \( m \) is odd and \( p = s_m + 1 \), then \( \alpha(C_p) = m - 1 \). In all other cases, \( \alpha(C_p) = m \).

Corollary 2.18. \( \alpha(\text{cor}(G) \times \text{cor}(H)) \geq \alpha(C_{2(p_1 + p_2)}) \) provided \( G \times H \) is hamiltonian of even order for any graphs \( G \) and \( H \) with \( |V(G)| = p_1 \) and \( |V(H)| = p_2 \).

Corollary 2.19. \( \alpha(C_p \times K_2) \geq \alpha(C_{2p}) \).

Theorem 2.20. If \( G \) is a finite graph with \( p \geq 3 \) and degree sequence \( d_1 \leq d_2 \leq \cdots \leq d_p \) such that \( d_i \leq i < p/2 \Rightarrow d_{p-i} \geq n - i \) then \( \psi(G) \geq \psi(C_p) \) and \( \alpha(G) \geq \alpha(C_p) \).

Theorem 2.21. If \( G \) is a self complementary graph of order \( p \), then \( \psi(G) \geq \psi(C_p) \) and \( \alpha(G) \geq \alpha(C_p) \).

Theorem 2.22. If \( \psi(G) = a \) and \( \psi(H) = b \), then \( \psi(G \times H) \geq \psi(K_a \times K_b) \).

Theorem 2.23. Let \( P_n^r = P_n \times \cdots \times P_n, r \)-times the cartesian product of \( P_n \), the path on \( n \) vertices. Then \( \psi(P_n^r) \geq \psi(P_n) \).

Corollary 2.24. \( \alpha(P_n^r) \geq \alpha(P_n) \).
Theorem 2.25. \( \chi(P(n, k)) = \alpha(P(n, k)) = 2 \) and \( \psi(P(n, k)) \leq 2n+1 \) if \( n \) is even and \( k \) is odd.

Theorem 2.26. \( \alpha(P(n, k)) \geq \alpha(C_2n) \) and \( \psi(P(n, k)) \geq \psi(C_2n) \) when \( n \not\equiv 0 \mod 4 \) and \( k \neq n/2 \).

Problem 2.27. Find the achromatic and pseudochromatic number of \( P(n, k) \) when 1) \( n \) is odd, \( k \) is odd 2) \( n \) is even, \( k \) is even 3) \( n \) is odd and \( k \) is even.

3 Proofs

Proof of Theorem 2.3
Let \( G \) be the permutation graph of \( K_{1,n} \) with respect to the identity permutation. Then we can write \( V(G) = V(K_{1,n}^1) \cup V(K_{1,n}^2) \) and \( E(G) = E(K_{1,n}^1) \cup E(K_{1,n}^2) \cup \{(u,v) : u \in V(K_{1,n}^1), v \in V(K_{1,n}^2) \) and \( v = I(u) \), where \( I \) is the identity permutation on the vertices of \( K_{1,n} \). That is, if \( V(K_{1,n}^1) = \{u; u_1, u_2, \ldots, u_n\} \) and \( V(K_{1,n}^2) = \{v; v_1, v_2, \ldots, v_n\} \) then \( E(G) = \{(u, u_i), (v, v_i) \) and \( (u, v), (u_i, v_i) \}: 1 \leq i \leq n \} \) as \( I(u) = v \) and \( I(u_i) = v_i \).

Note that any pseudoachromatic coloring \( \psi \) of \( G \) must assign the colors to \((u_i, v_i)\) only in pairs. So it follows that \( \left(\frac{\psi}{2}\right) \leq n \). That is, \( \psi(\psi - 1) \leq 2n \) and \( \psi \leq (1 + \sqrt{1 + 8n})/2 \). If \( \psi = (1 + \sqrt{1 + 8n})/2 \) then it is a reality to exhibit a \((\psi + 1)\) pseudoachromatic coloring as follows: \( \psi(u) = \psi + 1; \psi(v) = \psi + 1; \psi(u_i) = 1, \) for \( i = 1, \ldots, x - 1; \psi(v_i) = i + 1, \) for \( i = 1, \ldots, x - 1; \psi(u_i) = 2, \psi(v_i) = i - (x - 3), \) for \( i = x, \ldots, 2x - 3; \psi(u_i) = 3, \psi(v_i) = i - (2x - 6), \) for \( i = 2x - 2, \ldots, 3x - 6; \psi(u_i) = 4, \psi(v_i) = i - (3x - 10), \) for \( i = 3x - 5, \ldots, 4x - 10; \) etc., \( \psi(u_i) = k, \psi(v_i) = i - \left(\left(k - 1\right)x - \left(\frac{k + 1}{2}\right)\right) \), for \( i = (k - 1)x - \left(\left(k - \frac{k + 1}{2}\right)\right) + 1, \ldots, \left(kx - \left(\frac{k + 1}{2}\right)\right) \); \( \psi(u_i) = x - 2, \psi(v_i) = i - \left((x - 3)x - \left(\left(x - 1\right)x - \left(\frac{x - 1}{2}\right)\right)\right), \) for \( i = (x - 3)x - \left(\left(x - 1\right)x - \left(\frac{x - 1}{2}\right)\right) + 1, \ldots, (x - 2)x - \left(\left(x - 1\right)x - \left(\frac{x - 1}{2}\right)\right) \); \( \psi(u_i) = x - 1, \psi(v_i) = i - \left((x - 2)x - \left(\frac{x}{2}\right)\right) \), for \( i = (x - 2)x - \left(\left(x - 1\right)x - \left(\frac{x}{2}\right)\right) + 1, \ldots, (x - 1)x - \left(\frac{x}{2}\right). \)\]

Proof of Theorem 2.4
Let \( G \) be the graph as defined in Theorem 2.3. Let \( \alpha \) be any permutation of the vertices of one copy of \( K_{1,n} \). Note that any pseudoachromatic coloring \( \psi \) of \( G \) must assign the colors to \((u_i, v_{\alpha(i)})\) only in pairs. So \( \psi \leq (1 + \sqrt{1 + 8n})/2 \). A \( \psi \)-coloring of \( G \) can be exhibited exactly as in Theorem 2.4. Then \( \alpha(\psi + 1)^{th} \)
color can be assigned to the vertices \( u \) and \( v \) and the result follows. \( \square \)

**Proof of Theorem 2.5**

Let \( G \) be the permutation graph of \( K_{m,n} \) with respect to any permutation \( \alpha \). Then we can write \( V(G) = V(K_{m,n}^1) \cup V(K_{m,n}^2) \) and \( E(G) = E(K_{m,n}^1) \cup E(K_{m,n}^2) \cup \{(u,v) : u \in V(K_{m,n}^1), v \in V(K_{m,n}^2) \} \), where \( \alpha \) is a permutation on the vertices of \( K_{m,n} \). Now let \( V(K_{m,n}^1) = A \) and \( B = \{v_1, \ldots, v_m\}; V(K_{m,n}^2) = C \) and \( D = \{v'_1, \ldots, v'_n\} \). Let \( \zeta \) be any coloring of \( G \) as follows. \( \zeta(u_i) = i \), for \( 1 \leq i \leq m; \zeta(v_i) = i + 1 \), for \( 1 \leq i \leq n \); \( \zeta(w_i) = m + i + 1 \), for \( 1 \leq i \leq n - m \); \( \zeta(u'_i) = m + 1 \); \( \zeta(u'_i) = m + i + 1 \), for \( 2 \leq i \leq n - m \); \( \zeta(w'_i) = m + i + 1 \), for \( 1 \leq i \leq n - m \); \( \zeta(v'_a) = a \) where \( a \) is any color in \( \{1, \ldots, n + 1\} \), for \( 1 \leq j \leq m \).

Then it is easy to check that \( \zeta \) is a pseudoachromatic coloring of \( G \) with \( n + 1 \) colors. Therefore \( |\zeta| = \psi \geq n + 1 \). We now claim that \( \psi \leq n + 1 \). Suppose that \( \psi = n + 2 \). Clearly all \((n + 2)\) colors cannot be present fully in \( A \) or \( B \) or \( C \) or \( D \). As \(|B| = |D| = n\), without loss of generality we assume that the colors of \( B \) are all distinct. Then the \((n + 1)^{th}\) and the \((n + 2)^{th}\) color cannot only be present in \( A \) or \( C \) or \( D \). So they have to appear at least once in the following manner.

I: \((n + 1)^{th}\) color in \( A \) and \((n + 2)^{th}\) color in \( B \);

II: \((n + 1)^{th}\) color in \( A \) and \((n + 2)^{th}\) color in \( D \);

III: \((n + 1)^{th}\) color in \( C \) and \((n + 2)^{th}\) color in \( D \).

**Case 1.** It happens.

Let \( u_1 \) be the vertex with color \( n + 1 \) and \( u'_1 \) is the vertex with color \( n + 2 \). Assume that the distinct colors of \( B \) are the colors \( 1 \) to \( n \). Now to ensure the existence of at least one edge between each of \((n + 1)(n + 2)/2\) color pairs, we are now left with \((m - 1)\) vertices in \( A \) and \( C \) and \( n \) vertices in \( D \). The \((m - 1)\) vertices in \( A \) can account for \( m(m + 1)/2 \) color pairs: \((i,j)\) for \( j = 2 \) to \( m + 1 \) by fixing \( i \) and then varying \( i \) for \( i = 1 \) to \( m \). So assign the colors \( m + 2 \) to \( n \) to the remaining \((m - 1)\) vertices of \( C \). Now as we have to have at least one edge between \((1, m + 2), (1, m + 3), \ldots, (1, n), (2, m + 2), (2, m + 3), \ldots, (2, n), \ldots, (n, m + 2), (n, m + 3), \ldots, (n, n + 1)\), we should compulsorily allot to the \( n \)-vertices of \( D \), the colors \( 1 \) to \( n \). This accounts for all the \((n + 1)(n + 2)/2\) color pairs provided \((u_1, u'_1) \in E(G)\). Else, we derive a contradiction.

The cases II and III can be dealt with on similar lines. Hence we have. \( \square \)

**Proof of Theorem 2.6**

Let \( V(G) = \{u_1, \ldots, u_p\} \). Let \( V(G \circ K_{1,1}) = V(G) \cup \{u'_1, u'_2, \ldots, u'_p\} \) and \( E(G \circ K_{1,1}) = E(G) \cup \{(u_i, u'_i) : 1 \leq i \leq p\} \). Let \( \zeta \) be any pseudoachromatic coloring of \( G \circ K_{1,1} \). Let \( \zeta(V(G \circ K_{1,1})) = \zeta(V(G)) \) and \( \zeta(u'_i) = \psi + 1 \) for
1 \leq i \leq p$. Then it is easy to check that $\zeta$ is a pseudoachromatic $(\psi+1)$-coloring of $G \circ K_{1,1}$ and hence $\psi(G \circ K_{1,1}) \geq \psi+1$. We claim that $\psi(G \circ K_{1,1}) = \psi+1$. Suppose $\zeta$ uses $\psi+2$ colors, then the $(\psi+2)th$ color has to be adjacent with the $\psi$-colors of $G$ and the $(\psi+1)th$ color of the vertices $u'_i$. Suppose that $\zeta$ has a few redundant vertices where one of the $\psi$-colors is used. We can recolor any such vertex with the $(\psi+2)th$ color. But as the maximum number of colors used in a pseudoachromatic coloring of $G$ is $\psi$, $(\psi+2)th$ color must have non-adjacency with some of the $\psi$-colors of $G$. Those colors cannot be assigned to the vertices $u'_i$ and hence results in at least one pair of colors in which $(\psi+2)th$ color is present with no edge between them, a contradiction. Therefore $|\zeta| \leq \psi+1$ and hence $\psi(G \circ K_{1,1}) = \psi+1$. \hfill \Box

Proof of Theorem 2.7
Let $V(B_{n,n}) = \{u; u_1, \ldots, u_n; v; v_1, \ldots, v_n\}$ and $E(B_{n,n}) = \{(u, u_i), (u, v), (v, v_i) : 1 \leq i \leq n\}$. Define $\psi : V(B_{n,n}) \rightarrow \{1, 2, 3\}$ as follows: $\psi(u) = 1$, $\psi(v) = 2$, $\psi(u_1) = 3$, $\psi(v_1) = 3$ and assign to the remaining vertices any one of these three colors. It is easy to see that $\psi$ is a pseudoachromatic coloring of $B_{n,n}$ and $\psi(B_{n,n}) \geq 3$. We claim that $\psi(B_{n,n}) \leq 3$. Suppose not and $\psi(B_{n,n}) = 4$. Since $\psi$ is an optimal pseudoachromatic coloring and $\deg(u) = \deg(v) = n$, $\psi$ should assign distinct colors to $u$ and $v$. Again as $\deg(u_i) = \deg(v_i) = 1$, the remaining two colors that appear among them has no edge between them, a contradiction. \hfill \Box

Proof of Corollary 2.8
Follows from Proposition 2.7. \hfill \Box

Proof of Corollary 2.13
Note that $C_p \times K_2$ is hamiltonian. Hence it has a hamiltonian cycle of length $C_{2p}$ as an induced graph. Thus $\psi(C_p \times K_2) \geq \psi(C_{2p})$ as $\psi$ is a monotone function. \hfill \Box

Proof of Proposition 2.14
$J(m, n)$, the Jelly fish graph is defined as follows:
$V(J_{m,n}) = \{u, v, w, x\} \cup \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ and $E(J_{m,n}) = \{(u, u_i), 1 \leq i \leq m; (v, v_j), 1 \leq j \leq n; (u, x), (u, w), (w, v), (w, x), (x, v)\}$. Define $\psi : V(J_{m,n}) \rightarrow \{1, 2, 3, 4\}$ as follows: $\psi(u) = 1$, $\psi(w) = 2$, $\psi(x) = 3$, $\psi(v) = 4$; $\psi(u_1) = 4$ and assign for the rest of the vertices any of these four colors. Then it is easy to check that $\psi$ is a pseducomplete coloring and hence $\psi(J_{m,n}) \geq 4$. Suppose that $J(m, n) = 5$ under some optimal pseudoachromatic coloring $\zeta$. Obviously $\zeta$ will assign distinct colors to the higher degree vertices. Therefore the colors on $u, v, w$ and $x$ must be distinct. Now the fifth color must appear among the pendant vertices of $J_{m,n}$. If that is the case, then
as the vertices $x$ and $w$ are not adjacent with any of the pendant vertices of $J_{m,n}$, there exists at least one color pair with no edge between them in $J_{m,n}$, a contradiction. \hfill $\Box$

**Proof of Theorem 2.15**

First we claim that $cor(G) \times cor(H)$ is hamiltonian. Let $u^*$ be the vertex adjacent to any vertex $u$ of $G$ or $H$ in the corona. Note that the set $\{(u,v), (u^*,v), (u,v^*), (u^*,v^*)\}$ induces a cycle of length 4 in $cor(G) \times cor(H)$ for each $u \in V(G)$ and $v \in V(H)$. Hence decompose the vertex set of $cor(G) \times cor(H)$ into such sets of 4 elements each. Without loss of generality let $(u_1,v)$ and $(u_2,v)$ be the first two vertices of a hamiltonian cycle $C$ of $G \times H$. We can construct a hamiltonian cycle $C'$ of $cor(G) \times cor(H)$ that begins its traversal of the first above said set of 4 elements at $(u_1,v)$ and stops at $(u_1,v^*)$. Then it visits $(u_2,v^*)$ and completes the first set visit at $(u_2,v)$. Upon repeatedly doing this process, the hamiltonian cycle $C'$ exhausts all the visits of all such sets of 4-elements for each vertex of $C$. Then it revisits the first set of 4-elements. At this point when we note that $C$ is of even order, the revisit of $C'$ occurs at $(u_1,v)$ to subsequently qualify the claim. As $|V(cor(G) \times cor(H))| = 2(p_1 + p_2)$, there exists a hamiltonian cycle of length $2(p_1 + p_2)$ in $cor(G) \times cor(H)$. Now as $C_{2(p_1+p_2)}$ is a subgraph of $cor(G) \times cor(H)$ and $\psi$ is monotone, we have $\psi(C_{2(p_1+p_2)}) \leq \psi(cor(G) \times cor(H))$. \hfill $\Box$

**Proof of Theorem 2.16**

We know that if $G$ is a self complementry graph of order $p$, then, as $|E(G)| = |E(G^c)| = \binom{p}{2} = p(p-1)/4$, $p = 4k$ or $4k+1$. Now if $\zeta$ is a pseudoachromatic $\psi$-coloring of $G$ then $\binom{\psi}{2} \leq |E(G)|$ and the result follows. \hfill $\Box$

**Proof of Theorem 2.20**

Chvatal [7] showed that any graph $G$ with the hypothesis as in the theorem will contain a hamiltonian cycle and hence the result follows using Theorem 2.11 and Theorem 2.17. \hfill $\Box$

**Proof of Theorem 2.21**

Note that the degree sequence of $G$ satisfies $d_i \leq i - 1 < \frac{p+1}{2} \Rightarrow d_{p+i} \geq p - i$ and hence Clapham [8] and Camion [9] have showed independently that $G$ has a hamiltonian cycle by observing that if a vertex $v$ is added to $G$ and when it is joined to all the vertices of $G$ then it satisfies the conditions of Chvatal’s theorem. Hence the result follows using Theorem 2.11 and Theorem 2.17. \hfill $\Box$

**Proof of Theorem 2.22**

Consider a pseudoachromatic a-coloring for $G$ and a pseudoachromatic $b$-
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coloring for $H$. Let $(c_1, \ldots, c_a)$ and $(c'_1, \ldots, c'_b)$ be the color classes of $G$ and $H$. Then we can decompose the vertex set of $G \times H$ into sets $c_i \times c'_1, \ldots, c_a \times c'_b$ where the sets may or may not be independent. Now if $g$ is a pseudoachromatic coloring of $K_a \times K_b$, then one can assign to the vertices in $c_i \times c'_j$ the color $g(i, j)$, for $1 \leq i \leq a$, $1 \leq j \leq b$. It is easy to see that this result in a pseudoachromatic $\psi(K_a \times K_b)$-coloring for $G \times H$, and hence $\psi(G \times H) \geq \psi(K_a \times K_b)$.

Proof of Theorem 2.23

Assume that the graph $P_{n}^{r-1}$ has hamiltonian path by the principle of mathematical induction. Then there is a sequential arrangement of the vertices of $P_{n}^{r-1}$ namely $Z_{r-1} = u_1, \ldots, u_{n}^{r-1}$ corresponding to a hamiltonian path, where $u_i$ is the $(r-1)$ vector corresponding to the $i^{th}$ vertex in that order. Let $z_{r-1}^*$ be the reverse order of $z_{r-1}$. Now given any $(r-1)$-vector $u_\ell$, let $u_\ell * j$ be the $r$-vector obtained from $u_\ell$ by arranging the first $(r-1)$ coordinates of $u_\ell * j$ in the same way as that of $u_\ell$ and the last coordinate is $j$. Let $z_{r-1}^* * j = u_1 * j, \ldots, u_{n-1} * j$. Then one can verify that the path $z_{r-1}^* * 1, z_{r-1}^* * 2, z_{r-1}^* * 3, \ldots$ until we have either $z_{r-1}^* * n$ or $z_{r-1}^* * n$ relying on the parity of $n$, namely odd or even, is a hamiltonian path in $P_{n}^{r}$. Notice that $P_{n}^{r}$ has $n^r$ vertices. As this path is an induced subgraph on $P_{n}^{r}$, we have $\psi(P_{n}^{r}) \geq \psi(P_{n}^{r}).$

Proof of Theorem 2.25

We know that $P(n, k)$ is bipartite if $n$ is even and $k$ is odd [5, 13]. In view of this, the chromatic and achromatic number of $P(n, k)$ follows immediately. Furthermore, as $P(n, k) \subseteq K_{2n,2n}$ and $\psi$ is monotone, it follows from Theorem 2.1 that $\psi(P(n, k)) \leq \psi(K_{2n,2n}) = 2n + 1$.

Proof of Theorem 2.26

If $n \equiv 0 \pmod{4}$ and $k \equiv n/2$, then $P(n, k)$ is hamiltonian [2]. This means $C_{2n}$ is an induced subgraph of $P(n, k)$. Hence by Theorem 2.11 and the fact that $\psi$ is monotone it follows that $\psi(P(n, k)) \geq \psi(C_{2n})$.

For more on these colorings one can also refer to [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

4 Some Applications

4.1 Clique Minors and Pseudoachromatic Colorings

In a graph $G$ a clique minor is a set of connected subgraphs that are pairwise disjoint and pairwise adjacent. To construct large clique minors in odd-dimensional cliques, Wood in [10] used the notion of pseudoachromatic color-
ings. The only difference that separates a clique minor and pseudoachromatic coloring is that every color class is not required to be connected. Wood [10] showed that in a three-dimensional product, the color classes in a pseudoachromatic coloring can be made connected.

4.2 Pseudocolorings in Parallel Computing

The graph coloring task focus on allotment of colors to the vertices of a graph such that the vertices which are adjacent are colored differently. Normally the main objective is to minimize the number of colours used. In parallel computing tasks, a graph coloring is done to split the work endowed with the vertices into independent subtasks in such a way that the subproblems are handled with simultaneously. As the emphasis is on obtaining least number of colors, the other aspect namely the speed should not be compromised. When the magnitude of the work allotted with every vertex is less and the task of finding new colorings to be repeated again and again then the total time taken to do the colorings might consume a sizeable portion of the whole computation. In such circumstances it is worth to find a usable coloring quicker than wasting time on minimizing the number of colors used. In [4] the authors adopted the approach of reducing the total time taken to develop scalable parallel coloring algorithms based on greedy methods. According to them scalability of a parallel algorithm is the ability to increase the speed as the number of processors is increased for a given task size. The method is to perform first a parallel pseudocoloring of a graph. To rectify this they do a second parallel step to smell the inconsistencies. These are removed in the last sequential step. They generalized this notion and presented a second parallel procedure that involves lesser number of colors.

References


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