q-Extension of Tangent Numbers and Polynomials

Associated with the p-Adic q-Integral on \( \mathbb{Z}_p \)

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Abstract

In this paper we introduce the q-extension of tangent numbers \( T_{n,q} \) and polynomials \( T_{n,q}(x) \) associated with the p-adic q-integral on \( \mathbb{Z}_p \).

Some interesting results and relationships are obtained.

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1 Introduction

Throughout this paper, we always make use of the following notations: \( \mathbb{N} \) denotes the set of natural numbers and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \), \( \mathbb{C} \) denotes the set of complex numbers, \( \mathbb{Z}_p \) denotes the ring of p-adic rational integers, \( \mathbb{Q}_p \) denotes the field of p-adic rational numbers, and \( \mathbb{C}_p \) denotes the completion of algebraic closure of \( \mathbb{Q}_p \). Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-\nu(p)} = p^{-1} \). When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number \( q \in \mathbb{C} \), or p-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \) one normally assume that \( |q| < 1 \). If \( q \in \mathbb{C}_p \), we normally assume that \( |q - 1|_p < p^{-\frac{1}{p-1}} \) so that \( q^x = \exp(x \log q) \) for \( |x|_p \leq 1 \). Throughout this paper we use the notation:

\[
[x]_q = \frac{1 - q^x}{1 - q}.
\]
Hence, \( \lim_{q \to 1} [x] = x \) for any \( x \) with \( |x|_p \leq 1 \) in the present \( p \)-adic case. Let \( UD(Z_p) \) be the space of uniformly differentiable function on \( Z_p \). For \( g \in UD(Z_p) \) the fermionic \( p \)-adic invariant \( q \)-integral on \( Z_p \) is defined by Kim as follows:

\[
I_{-q}(g) = \int_{Z_p} g(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} g(x)(-q)^x, \text{ see [1] . (1.1)}
\]

If we take \( g_n(x) = g(x + n) \) in (1.1), then we see that

\[
q^n I_q(g_n) + (-1)^{n-1} I_q(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l). \quad (1.2)
\]

Let us define the tangent numbers \( T_n \) and polynomials \( T_n(x) \) as follows:

\[
\sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \frac{2}{e^{2t} + 1}, \quad \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!} = \left( \frac{2}{e^{2t} + 1} \right) e^{xt} (\text{see [2, 3]}). \quad (1.3)
\]

Numerous properties of tangent number are known. More studies and results in this subject we may see references [2], [3], [4], [5], [6]. About extensions for the tangent numbers can be found in [4, 5, 6]. Our aim in this paper is to define \( q \)-extension of tangent polynomials \( T_{n,q}(x) \) associated with the \( p \)-adic \( q \)-integral on \( Z_p \). We investigate some properties which are related to \( q \)-extension of tangent numbers \( T_{n,q} \) and polynomials \( T_{n,q}(x) \). We also derive the existence of a specific interpolation function which interpolate \( q \)-extension of tangent numbers \( T_{n,q} \) and polynomials \( T_{n,q}(x) \) at negative integers.

### 2 \( q \)-extension of tangent polynomials

Our primary goal of this section is to define \( q \)-extension of tangent numbers \( T_{n,q} \) and polynomials \( T_{n,q}(x) \). We also find generating functions of \( q \)-extension of tangent numbers \( T_{n,q} \) and polynomials \( T_{n,q}(x) \) and investigate their properties. For \( q \in C_p \) with \( |1 - q|_p \leq 1 \), if we take \( g(x) = e^{2xt} \) in (1.2), then we easily see that

\[
I_{-q}(e^{2xt}) = \int_{Z_p} e^{2xt} d\mu_{-q}(x) = \frac{[2]_q}{qe^{2t} + 1}.
\]

Let us define the \( q \)-extension of tangent numbers \( T_{n,q} \) and polynomials \( T_{n,q}(x) \) as follows:

\[
\int_{Z_p} e^{2yt} d\mu_{-q}(y) = \sum_{n=0}^{\infty} T_{n,q} \frac{t^n}{n!}, \quad (2.1)
\]

\[
\int_{Z_p} e^{(x+2y)t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!}. \quad (2.2)
\]

By (2.1) and (2.2), we obtain the following Witt’s formula.
**Theorem 2.1** For $n \in \mathbb{Z}_+$, we have

\[
\int_{\mathbb{Z}_p} (2x)^n d\mu_q(x) = T_{n,q} \quad \text{and} \quad \int_{\mathbb{Z}_p} (x + 2y)^n d\mu_q(y) = T_{n,q}(x).
\]

By using $p$-adic $q$-integral on $\mathbb{Z}_p$, we obtain,

\[
\int_{\mathbb{Z}_p} e^{2xt} d\mu_q(x) = [2]_q \sum_{m=0}^{\infty} (-1)^mq^m e^{2mt}.
\]

(2.3)

Thus $q$-extension of tangent numbers $T_{n,q}$ are defined by means of the generating function

\[
F_q(t) = \sum_{n=0}^{\infty} T_{n,q} \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^mq^m e^{2mt}.
\]

(2.4)

Using similar method as above, by using $p$-adic $q$-integral on $\mathbb{Z}_p$, we have

\[
\sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!} = \left( \frac{[2]_q}{qe^{2t} + 1} \right) e^{xt}.
\]

(2.5)

By using (2.2) and (2.5), we obtain

\[
F_q(t, x) = \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^mq^m e^{(2m+x)t}.
\]

(2.6)

By Theorem 2.1, we easily obtain that

\[
T_{n,q}(x) = \int_{\mathbb{Z}_p} (x + 2y)^n d\mu_q(y)
= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} T_{k,q}
= (x + T_q)^n
= [2]_q \sum_{m=0}^{\infty} (-1)^mq^m (x + 2m)^n.
\]

(2.7)

The following elementary properties of $q$-extension of tangent polynomials $T_{n,q}(x)$ are readily derived from (2.1) and (2.2). We, therefore, choose to omit the details involved. We note that

\[
\lim_{q \to 1} T_{n,q}(x) = T_n(x) \quad \text{and} \quad \lim_{q \to 1} T_{n,q} = T_n.
\]
Theorem 2.2 For any positive integer \(n\), we have
\[ T_{n,q}^{-1}(2 - x) = (-1)^n T_{n,q}(x). \]

Theorem 2.3 For any positive integer \(m\)(=odd), we have
\[ T_{n,q}(x) = \frac{[2]_q}{[2]_{q^m}} m^n \sum_{a=0}^{m-1} (-1)^a w^a q^a T_{n,q}^m \left( \frac{2a + x}{m} \right), \quad n \in \mathbb{Z}_+. \]

By (1.2), (2.1), and (2.2), we easily see that
\[ [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l (2l)^m = q^n T_{m,q}(2n) + (-1)^{n-1} T_{m,q}. \]

Hence, we have the following theorem.

Theorem 2.4 Let \(m \in \mathbb{Z}_+\). If \(n \equiv 0 \pmod{2}\), then
\[ q^n T_{m,q}(2n) - T_{m,q} = [2]_q \sum_{l=0}^{n-1} (-1)^{l+1} q^l (2l)^m. \]

If \(n \equiv 1 \pmod{2}\), then
\[ q^n T_{m,q}(2n) + T_{m,q} = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l (2l)^m. \]

From (1.1), we note that
\[ [2]_q = q \int_{\mathbb{Z}_p} e^{(2x+2)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} e^{2xt} d\mu_{-q}(x) \]
\[ = \sum_{n=0}^{\infty} \left( q \int_{\mathbb{Z}_p} (2x + 2)^n d\mu_{-q}(x) + \int_{\mathbb{Z}_p} (2x)^n d\mu_{-q}(x) \right) \frac{t^n}{n!} \]
\[ = \sum_{n=0}^{\infty} \left( qT_{n,q}(2) + T_{n,q} \right) \frac{t^n}{n!}. \]

Therefore, we obtain the following theorem.

Theorem 2.5 For \(n \in \mathbb{Z}_+\), we have
\[ qT_{n,q}(2) + T_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases} \]

By (2.7) and Theorem 2.5, we have the following corollary.
Corollary 2.6 For $n \in \mathbb{Z}_+$, we have

$$q(T_q + 2)^n + T_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention of replacing $(T_q)^n$ by $T_{n,q}$.

Theorem 2.7 For $n \in \mathbb{Z}_+$, we have

$$T_{n,q}(x + y) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q}(x)y^{n-l}.$$ 

By Theorem 2.1, we easily get

$$T_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \int_{\mathbb{Z}_q} (2y)^l d\mu_{-q}(y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l}T_{l,q}.$$ 

Therefore, we obtain the following theorem.

Theorem 2.8 For $n \in \mathbb{Z}_+$, we have

$$T_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q}x^{n-l}.$$ 

3 $q$-extension of tangent zeta function

In this section, by using $q$-extension of tangent numbers and polynomials, we give the definition for the $q$-extension of tangent zeta function and $q$-extension of Hurwitz-type tangent zeta functions. These functions interpolate the $q$-extension of tangent numbers and tangent polynomials, respectively. Let $q$ be a complex number with $|q| < 1$. From (2.4), we note that

$$\left. \frac{d^k}{dt^k} F_q(t) \right|_{t=0} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m (2m)^k = T_{k,q}, \ (k \in \mathbb{N}).$$ 

(3.1)

By using the above equation, we are now ready to define $q$-extension of tangent zeta functions.

Definition 3.1 For $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, define the $q$-extension of tangent zeta function by

$$\zeta_q(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(2n)^s}. \quad (3.2)$$
Note that \( \zeta_q(s) \) is a meromorphic function on \( \mathbb{C} \). Observe that if \( q \to 1 \), then \( \zeta_q(s) = \zeta_T(s) \), where \( \zeta_T(s) \) are tangent zeta functions (see [2]). Relation between \( \zeta_q(s) \) and \( T_{k,q} \) is given by the following theorem.

**Theorem 3.2** For \( k \in \mathbb{N} \), we obtain

\[
\zeta_q(-k) = T_{k,q}. \tag{3.3}
\]

Observe that \( \zeta_q(s) \) function interpolates \( T_{k,q} \) numbers at non-negative integers. By using (2.7), we note that

\[
\frac{d^k}{dt^k} F_q(t, x) \bigg|_{t=0} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m (x + 2m)^k = T_{k,q}(x), (k \in \mathbb{N}), \tag{3.4}
\]

and

\[
\left( \frac{d}{dt} \right)^k \left( \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!} \right) \bigg|_{t=0} = T_{k,q}(x), \text{ for } k \in \mathbb{N}. \tag{3.5}
\]

By (3.2), (3.4) and (3.5), we are now ready to define the \( q \)-extension of Hurwitz-type tangent zeta functions.

**Definition 3.3** For \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), \( q \)-extension of Hurwitz-type tangent zeta function by

\[
\zeta_q(s, x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(2n + x)^s}. \tag{3.6}
\]

Note that \( \zeta_q(s, x) \) is a meromorphic function on \( \mathbb{C} \). Observe that if \( q \to 1 \), then \( \zeta_q(s, x) = \zeta_T(s, x) \), where \( \zeta_T(s, x) \) are the Hurwitz-type tangent zeta functions (see [2]). Relation between \( \zeta_q(s, x) \) and \( T_{k,q}(x) \) is given by the following theorem.

**Theorem 3.4** For \( k \in \mathbb{N} \), we obtain

\[
\zeta_q(-k, x) = T_{k,q}(x). \]

**References**


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