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Generalized $\mu^{(m,n)}b$-Continuous Functions in Bigeneralized Topological Spaces

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Abstract

The purpose of this paper is to introduce the notions of $g\mu^{(m,n)}b$-continuous, almost $g\mu^{(m,n)}b$-continuous and regular strongly $g\mu^{(m,n)}b$-continuous functions in bigeneralized topological spaces. Basic properties, characterizations and relationships between these functions are obtained.

Keywords: bigeneralized topological spaces, $g\mu^{(m,n)}b$- continuous, almost $g\mu^{(m,n)}b$-continuous and regular strongly $g\mu^{(m,n)}b$-continuous functions

1 Introduction

Császár introduced the concepts of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. In particular, he investigated characterizations for the generalized continuous function by using a closure operator defined on generalized neighborhood systems. In [6], W. K. Min introduced the notion of almost (g, g')-continuity and investigated properties of such functions and relationships among (g, g')-continuity, almost (g, g')-continuity and weak

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(g, g')-continuity. In 2010, C. Boonpok [1] introduced the concepts of bigeneralized topological spaces and studied (m, n)-closed sets and (m, n)-open sets in bigeneralized topological spaces. He also introduced the notion of weakly open functions in bigeneralized topological spaces and investigated properties of such functions. Recently, in 2011, T. Duangphui, C. Boonpok, and C. Viriyapong [5] introduced the notions of $(\mu, \mu')^{(m,n)}$-continuous and almost $(\mu, \mu')^{(m,n)}$-continuous functions. They obtained many characterizations and properties of such functions.

In this paper the concepts of generalized $\mu^{(m,n)}b$- continuous (briefly $g\mu^{(m,n)}b$-continuous), almost $g\mu^{(m,n)}b$-continuous and regular strongly $g\mu^{(m,n)}b$-continuous (briefly rs-$g\mu^{(m,n)}b$-continuous) functions are defined and characterized.

2 Preliminaries

Definition 2.1 [3] Let X be a nonempty set. A collection μ of subsets of X is a *generalized topology* (or briefly GT) on X if it satisfies the following:

(O1) $\varnothing \in \mu$; and

(O2) If $\{M_i : i \in I\} \subseteq \mu$, then $\bigcup_{i \in I} M_i \in \mu$.

If μ is a GT on X, then (X, μ) is called a *generalized topological space* (or briefly GTS), and the elements of μ are called *μ-open* sets. A subset F of X is said to be *μ-closed* if the complement $X \backslash F$ of F is μ-open.

Definition 2.2 [2] Let (X, μ) be a GTS and let $A \subseteq X$. The *μ-interior* of A, denoted by $i_\mu(A)$, is the union of all μ-open sets contained in A. The *μ-closure* of A, denoted by $c_\mu(A)$, is the intersection of all μ-closed sets containing A.

Definition 2.3 [2] Let (X, μ) be a GTS and $A \subseteq X$. Then

(i) A is said to be *μb-open* if $A \subseteq c_\mu(i_\mu(A)) \cup i_\mu(c_\mu(A))$. A set is *$\mu b$-closed* if its complement is μb-open.

(ii) The *μb-closure* of A, denoted by $bc_\mu(A)$ is the intersection of all μb-closed sets containing A.

(iii) The *μb-interior* of A, denoted by $bi_\mu(A)$ is the union of all μb-open sets contained in A.

Definition 2.4 [7] Let (X, μ) be a GTS and $A \subseteq X$. Then

(i) A is said to be *generalized μb-open* (briefly $g\mu b$-open) if $U \subseteq A$ for some μ-closed set U, then $U \subseteq bi_\mu(A)$. The complement of a $g\mu b$-open set is called a *$g\mu b$-closed* set. Equivalently, A is $g\mu b$-closed if $A \subseteq U$ for some μ-open set U, then $bc_\mu(A) \subseteq U$.

(ii) The *$g\mu b$-closure* of A, denoted by $gbc_\mu(A)$, is the intersection of all $g\mu b$-closed sets containing A. Thus, $A \subseteq gbc_\mu(A)$.

(iii) The *$g\mu b$-interior* of A, denoted by $gbi_\mu(A)$, is the union of all $g\mu b$-open sets contained in A. Hence, $gbi_\mu(A) \subseteq A$.

The proofs of the following are straightforward.

Theorem 2.5 Let (X, μ) be a GTS and $A \subseteq X$.

(i) *If A is μ-open, then A is $g\mu b$-open.*

(ii) $i_\mu(A) \subseteq bi_\mu(A)$.

(iii) *If A is $g\mu b$-open, then $gbi_\mu(A) = A$.*

(iv) *If A is $g\mu b$-closed, then $gbc_\mu(A) = A$.*

(v) $i_\mu(A) = X \backslash c_\mu(X \backslash A)$

(vi) $gbi_\mu(A) = X \backslash gbc_\mu(X \backslash A)$

Theorem 2.6 [4] *Let (X, μ) be a GTS and $A \subseteq X$. Then $q \in gbc_\mu(A)$ if and only if for every $g\mu b$-open set O with $q \in O$, $O \cap A \neq \varnothing$.*

Definition 2.7 [1] Let X be a nonempty set and let μ_1, μ_2 be generalized topologies on X. The triple (X, μ_1, μ_2) is said to be a *bigeneralized topological space* (briefly $BGTS$).

Throughout this paper, m and n are elements of the set $\{1, 2\}$ where $m \neq n$. Let (X, μ_1, μ_2) be a $BGTS$ and A a subset of X. The closure of A and the interior of A with respect to μ_m are denoted by $c_{\mu_m}(A)$ and $i_{\mu_m}(A)$, respectively. The gb-closure of A and the gb-interior of A with respect to μ_m are denoted by $gbc_{\mu_m}(A)$ and $gbi_{\mu_m}(A)$, respectively.

3 $g\mu^{(m,n)}b$-Continuous functions

In this section, we introduce $g\mu^{(m,n)}b$-continuous, almost $g\mu^{(m,n)}b$-continuous and rs-$g\mu^{(m,n)}b$-continuous functions in bigeneralized topological spaces and study some of their properties and relationships..

All throughout this paper $(X, \mu_X^1, \mu^2{}_X)$ and (Y, μ_Y^1, μ_Y^2) are bigeneralized topological spaces.

Definition 3.1 A function $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ is said to be $g\mu^{(m,n)}b$-*continuous* if for each μ_Y^m-open set V, $f^{-1}(V)$ is $g\mu_X^n b$-open set in X.

Definition 3.2 A function $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ is said to be *pairwise $g\mu b$-continuous* if f is both $g\mu^{(1,2)}b$-continuous and $g\mu^{(2,1)}b$-continuous.

Example 3.3 Let $X = \{a, b, c\}$ and $Y = \{d, e\}$. Consider the generalized topologies $\mu_X^1 = \{\varnothing, \{a\}, \{b, c\}, X\}$, $\mu_X^2 = \{\varnothing, \{c\}, \{b, c\}\}$, $\mu_Y^1 = \{\varnothing, \{d\}\}$, and $\mu_Y^2 = \{\varnothing, \{e\}, Y\}$. All subsets of X are $g\mu_X^1 b$-open and the $g\mu_X^2 b$-open sets are $\varnothing, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}$ and X. Let $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ be a function defined by $f(a) = f(c) = d$, $f(b) = e$. Then f is $g\mu^{(1,2)}b$-continuous and $g\mu^{(2,1)}b$-continuous. Thus, f is pairwise $g\mu b$-continuous.

Theorem 3.4 *For a function $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$, the following properties are equivalent.*

(i) *For every $x \in X$ and for each μ_Y^m-open set V in Y containing $f(x)$, there exists a $g\mu_X^n b$-open set U containing x such that $f(U) \subseteq V$;*

(ii) *$x \in gbi_{\mu_X^n}(f^{-1}(V))$ for every $V \in \mu_Y^m$ containing $f(x)$;*

(iii) *$x \in gbi_{\mu_X^n}(f^{-1}(B))$ for every subset B of Y with $x \in f^{-1}(i_{\mu_Y^m}(B))$;*

(iv) *$x \in f^{-1}(F)$ for every μ_Y^m-closed subset F of Y such that $x \in gbc_{\mu_X^n}(f^{-1}(F))$.*

Proof: Let $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ be a function and $x \in X$.
(i)$\Rightarrow$ (ii): Let V be any μ_Y^m-open subset of Y such that $f(x) \in V$. By (1), there exists a $g\mu_X^n b$-open set $U \subseteq X$ with $x \in U$ and $f(U) \subseteq V$. Hence, $U \subseteq f^{-1}(V)$ with $x \in U$. This implies that $x \in gbi_{\mu_X^n}(f^{-1}(V))$.
(ii)$\Rightarrow$ (i): Let $V \in \mu_Y^m$ with $f(x) \in V$. By (ii), $x \in gbi_{\mu_X^n}(f^{-1}(V))$. Hence, there exists a $g\mu_X^n b$-open set $U \subseteq f^{-1}(V)$ such that $x \in U$.
(ii)$\Rightarrow$ (iii):Let $B \subseteq Y$ with $x \in f^{-1}(i_{\mu_Y^m}(B))$. Then $f(x) \in i_{\mu_Y^m}(B)$. Since $i_{\mu_Y^m}(B) \in \mu_Y^m$, by (ii), $x \in gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(B))) \subseteq gbi_{\mu_X^n}(f^{-1}(B))$. Thus, $x \in gbi_{\mu_X^n}(f^{-1}(B))$.
(iii)$\Rightarrow$ (iv): Let F be a μ_Y^m-closed subset of Y such that $x \notin f^{-1}(F)$. Then $x \in X \backslash f^{-1}(F) = f^{-1}(Y \backslash F) = f^{-1}(i_{\mu_Y^m}(Y \backslash F))$ since $Y \backslash F$ is μ_Y^m-open. By (iii),

$x \in gbi_{\mu_X^n}(f^{-1}(Y\backslash F))$=$gbi_{\mu_X^n}(X\backslash f^{-1}(F))$=$X\backslash gbc_{\mu_X^n}(f^{-1}(F))$. Hence, we have $x \notin gbc_{\mu_X^n}(f^{-1}(F))$.
(iv)$\Rightarrow$ (ii): Let $V \in \mu_Y^m$ with $f(x) \in V$. Suppose that $x \notin gbi_{\mu_X^n}(f^{-1}(V))$. Then $x \in X\backslash gbi_{\mu_X^n}(f^{-1}(V)) = gbc_{\mu_X^n}(X\backslash f^{-1}(V)) = gbc_{\mu_X^n}(f^{-1}(Y\backslash V))$. By (iv), we have $x \in f^{-1}(Y\backslash V) = X\backslash f^{-1}(V)$. This implies that $x \notin f^{-1}(V)$ which is a contradiction since $f(x) \in V$. Therefore, $x \in gbi_{\mu_X^n}(f^{-1}(V))$. □

Theorem 3.5 *If f is $g\mu^{(m,n)}b$-continuous, then for every $x \in X$ and for every μ_Y^m-open set V containing $f(x)$, there exists a $g\mu_X^n b$-open set U containing x such that $f(U) \subseteq V$.*

Proof: Let $x \in X$ and let V be any μ_Y^m-open set in Y with $f(x) \in V$. Since f is $g\mu^{(m,n)}b$-continuous, $f^{-1}(V)$ is $g\mu_X^n b$-open in X. Take $U = f^{-1}(V)$. Then there exist a $g\mu_X^n b$-open set U with $x \in U$ and $f(U) \subseteq V$. □

Remark 3.6 *The converse of Theorem 3.5 is not true.*

To see this consider the BGTS spaces (X, μ_X^1, μ_X^2) and (Y, μ_Y^1, μ_Y^2) in Example 3.3. Define $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ by $f(a) = f(b) = d$, $f(c) = e$. Then for $a \in X$ where $f(a) \in \{d\} \in \mu_Y^1$, there exists a $g\mu_X^2 b$-open set $\{a\}$ and $f(\{a\}) = \{d\}$. But f is not $g\mu^{(1,2)}b$-continuous.

Lemma 3.7 *Let (X, μ_X^1, μ_X^2) and (Y, μ_Y^1, μ_Y^2) be BGTS. Then the following are equivalent:*

(i) *$f^{-1}(V) = gbi_{\mu_X^n}(f^{-1}(V))$ for every $V \in \mu_Y^m$;*

(ii) *$f^{-1}(i_{\mu_Y^m}(B)) \subseteq gbi_{\mu_X^n}(f^{-1}(B))$ for every $B \subseteq Y$;*

(iii) *$gbc_{\mu_X^n}(f^{-1}(F) = f^{-1}(F)$ for every μ_Y^m-closed subset F of Y.*

Proof: (i)$\Rightarrow$ (ii): Let $B \subseteq Y$. Since $i_{\mu_Y^m}(B)) \in \mu_Y^m$, by (i), $f^{-1}(i_{\mu_Y^m}(B)) = gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(B))) \subseteq gbi_{\mu_X^n}(f^{-1}(B))$.

(ii)$\Rightarrow$ (iii): Let F be a μ_Y^m-closed subset of Y such that $x \notin f^{-1}(F)$. Then $x \in X\backslash f^{-1}(F) = f^{-1}(Y\backslash F) = f^{-1}(i_{\mu_Y^m}(Y\backslash F))$. By (ii), $x \in gbi_{\mu_X^n}(f^{-1}(Y\backslash F)) = gbi_{\mu_X^n}(X\backslash f^{-1}(F)) = X\backslash gbc_{\mu_X^n}(f^{-1}(F))$. Hence, $x \notin gbc_{\mu_X^n}(f^{-1}(F))$. Therefore, $gbc_{\mu_X^n}(f^{-1}(F)) \subseteq f^{-1}(F) \subseteq gbc_{\mu_X^n}(f^{-1}(F))$. It follows that $f^{-1}(F) = gbc_{\mu_X^n}(f^{-1}(F))$.

(iii)$\Rightarrow$ (i): Let $V \in \mu_Y^m$. Then $Y\backslash V$ is a μ_Y^m-closed set in Y. By (iii), $gbc_{\mu_X^n}(f^{-1}(Y\backslash V) = f^{-1}(Y\backslash V) = X\backslash f^{-1}(V)$. Thus, $X\backslash gbi_{\mu_X^n}(f^{-1}(V)) = X\backslash f^{-1}(V)$. Hence, $f^{-1}(V) = gbi_{\mu_X^n}(f^{-1}(V))$. □

Theorem 3.8 *Let (X, μ_X^1, μ_X^2) and (Y, μ_Y^1, μ_Y^2) be BGTS. For a $g\mu^{(m,n)}b$-continuous function $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$, the following properties hold.*

(i) $f^{-1}(V) = gbi_{\mu_X^n}(f^{-1}(V))$ *for every* $V \in \mu_Y^m$*;*

(ii) $f^{-1}(i_{\mu_Y^m}(B)) \subseteq gbi_{\mu_X^n}(f^{-1}(B))$ *for every* $B \subseteq Y$*;*

(iii) $gbc_{\mu_X^n}(f^{-1}(F) = f^{-1}(F)$ *for every* μ_Y^m*-closed subset* F *of* Y*.*

Proof: Let $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ be a $g\mu^{(m,n)}b$-continuous function.
(i): Let $V \in \mu_Y^m$. Since f is $g\mu^{(m,n)}b$-continuous $f^{-1}(V)$ is $g\mu_X^n b$-open in X. Hence, $f^{-1}(V) = gbi_{\mu_X^n}(f^{-1}(V))$.
(ii) and (iii): Follow immediately from Lemma 3.7. □

Remark 3.9 *As a consequence of Theorems 3.4, 3.5, and Lemma 3.7, the converse of Theorem 3.8 is not true.*

Definition 3.10 A function $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ is said to be *almost* $g\mu^{(m,n)}b$*-continuous at* $x \in X$ if for each μ_Y^m-open set V containing $f(x)$, there exists a $g\mu_X^n b$-open set U containing x such that $f(U) \subseteq i_{\mu_Y^m}(c_{\mu_Y^n}(V))$.

Definition 3.11 A function $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ is said to be *almost* $g\mu^{(m,n)}b$*-continuous* if f is almost $g\mu^{(m,n)}b$-continuous at every $x \in X$. It is *pairwise almost* $g\mu b$*-continuous* if it is almost $g\mu^{(1,2)}b$-continuous and almost $g\mu^{(2,1)}b$-continuous.

Theorem 3.12 *If* $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ *is* $g\mu^{(m,n)}b$*-continuous, then* f *is almost* $g\mu^{(m,n)}b$*-continuous.*

Proof: Let $x \in X$ and V be any μ_Y^m-open set in Y with $f(x) \in V$. Then $f^{-1}(V)$ is $g\mu_X^n b$-open in X and $x \in f^{-1}(V)$. Let $U = f^{-1}(V)$. Then $f(U) \subseteq V \subseteq i_{\mu_Y^m}(c_{\mu_Y^n}(V))$. Therefore, f is almost $g\mu^{(m,n)}b$-continuous. □

Remark 3.13 *The converse of Theorem 3.12 is not true.*

To see this let $X = \{a, b, c\} = Y$, $\mu_X^1 = \{\varnothing, \{a\}, \{b, c\}, X\}$, $\mu_X^2 = \{\varnothing, \{a, b\}, \{b, c\}, X\}$, $\mu_Y^1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ and $\mu_Y^2 = \{\varnothing, \{c\}, \{a, c\}\}$. Then all subsets of X are $g\mu_X^1 b$-open sets and $g\mu_X^2 b$-open sets of X are $\varnothing, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}$ and X. Let $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ be the identity function. Then f is almost $g\mu^{(1,2)}b$-continuous but it is not $g\mu^{(1,2)}b$-continuous.

Definition 3.14 Let (X, μ_X^1, μ_X^2) be a BGTS. Then $A \subseteq X$ is $\mu^{(m,n)}$*-regular open* if $A = i_{\mu_X^m}(c_{\mu_X^n}(A))$.

Theorem 3.15 *For a function* $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$*, the following properties are equivalent:*

(i) f *is almost* $g\mu^{(m,n)}b$*-continuous at* $x \in X$*;*

(ii) $x \in gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))))$ *for every* $V \in \mu_Y^m$ *containing* $f(x)$*;*

(iii) $x \in gbi_{\mu_X^n}(f^{-1}(V))$ *for every* $\mu^{(m,n)}$*-regular open subset* V *of* Y *containing* $f(x)$*;*

(iv) *For every* $\mu^{(m,n)}$*-regular open set* V *containing* $f(x)$*, there exists a* $g\mu_X^n b$*-open set* U *containing* x *such that* $f(U) \subseteq V$*.*

Proof: (i)⇒(ii): Let $V \in \mu_Y^m$ with $f(x) \in V$. Then $x \in f^{-1}(V)$. Since f is almost $g\mu^{(m,n)}b$-continuous at x, there exists a $g\mu_X^n b$-open set U containing x such that $f(U) \subseteq i_{\mu_Y^m}(c_{\mu_Y^n}(V))$. Thus, $x \in U \subseteq f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V)))$. Since U is a $g\mu_X^n b$-open set, we have $x \in gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))))$.

(ii)⇒(iii): Let V be any $\mu^{(m,n)}$-regular open subset of Y containing $f(x)$. Then $f(x) \in V = i_{\mu_Y^m}(c_{\mu_Y^n}(V))$. Since $V \in \mu_Y^m$, by (ii),

$$x \in gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V)))) = gbi_{\mu_X^n}(f^{-1}(V)).$$

Hence, $x \in gbi_{\mu_X^n}(f^{-1}(V))$.

(iii)⇒(iv): Let V be any $\mu^{(m,n)}$-regular open subset of Y containing $f(x)$. Then by (iii) $x \in gbi_{\mu_X^n}(f^{-1}(V))$. Hence, there exists a $g\mu_X^n b$-open set U with $x \in U \subseteq f^{-1}(V)$. Thus, $f(U) \subseteq V$.

(iv)⇒(i): Let $x \in X$ and V be a μ_Y^m-open set in Y containing $f(x)$. Then $f(x) \in V = i_{\mu_Y^m}(V) \subseteq i_{\mu_Y^m}(c_{\mu_Y^n}(V))$. Since $i_{\mu_Y^m}(c_{\mu_Y^n}(V))$ is $\mu^{(m,n)}$-regular open, by (iv), there exists a $g\mu_X^n b$-open set U with $x \in U$ and $f(U) \subseteq i_{\mu_Y^m}(c_{\mu_Y^n}(V))$, Therefore, f is almost $g\mu^{(m,n)}b$-continuous at $x \in X$. □

Theorem 3.16 *For a function* $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$*, the following properties are equivalent:*

(i) f *is almost* $g\mu^{(m,n)}b$*-continuous;*

(ii) $f^{-1}(V) \subseteq gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))))$ *for every* $V \in \mu_Y^m$*;*

(iii) $gbc_{\mu_X^n}(f^{-1}(c_{\mu_Y^m}(i_{\mu_Y^n}(F)))) \subseteq f^{-1}(F)$ *for every* μ_Y^m*-closed set* F *in* Y*;*

(iv) $gbc_{\mu_X^n}(f^{-1}(c_{\mu_Y^m}(i_{\mu_Y^n}(c_{\mu_Y^m}(B))))) \subseteq f^{-1}(c_{\mu_Y^m}(B))$ *for every* $B \subseteq Y$*;*

(v) $f^{-1}(i_{\mu_Y^m}(B)) \subseteq gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(i_{\mu_Y^m}(B)))))$ *for every* $B \subseteq Y$*;*

(vi) $f^{-1}(V) = gbi_{\mu_X^n}(f^{-1}(V))$ *for every* $\mu^{(m,n)}$*-regular open subset* V *of* Y*.*

(vii) $f^{-1}(F) = gbc_{\mu_X^n}(f^{-1}(F))$ *for every* $\mu^{(m,n)}$*-regular closed subset* F *of* Y*.*

Proof: (i)$\Rightarrow$ (ii): Let $V \in \mu_Y^m$ and $x \in f^{-1}(V)$. Then $f(x) \in V$. Since f is almost $g\mu^{(m,n)}b$-continuous, there exists a $g\mu_X^n b$-open set U with $x \in U$ and $f(U) \subseteq i_{\mu_Y^m}(c_{\mu_Y^n}(V))$. Hence, $x \in U \subseteq f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))$. Thus, $x \in gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))))$. Therefore, $f^{-1}(V) \subseteq gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))))$.

(ii)$\Rightarrow$ (iii): Let F be any μ_Y^m-closed set in Y. Then $Y\backslash F$ is a μ_Y^m-open set in Y. By (ii),

$$\begin{aligned} X\backslash f^{-1}(F) = f^{-1}(Y\backslash F) &\subseteq gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(Y\backslash F)))) \\ &= gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(Y\backslash i_{\mu_Y^n}(F)))) \\ &= gbi_{\mu_X^n}(f^{-1}(Y\backslash c_{\mu_Y^m}(i_{\mu_Y^n}(F)))) \\ &= gbi_{\mu_X^n}(X\backslash f^{-1}(c_{\mu_Y^m}(i_{\mu_Y^n}(F)))) \\ &= X\backslash gbc_{\mu_X^n}(f^{-1}(c_{\mu_Y^m}(i_{\mu_Y^n}(F)))). \end{aligned}$$

Hence, $gbc_{\mu_X^n}(f^{-1}(c_{\mu_Y^m}(i_{\mu_Y^n}(F)))) \subseteq f^{-1}(F)$.

(iii)$\Rightarrow$ (iv): Let $B \subseteq Y$. Since $c_{\mu_Y^m}(B)$ is μ_Y^m-closed in Y, by (iii),

$$gbc_{\mu_X^n}(f^{-1}(c_{\mu_Y^m}(i_{\mu_Y^n}(c_{\mu_Y^m}(B))))) \subseteq f^{-1}(c_{\mu_Y^m}(B)).$$

(iv)$\Rightarrow$ (v): Let $B \subseteq Y$. By Theorem 2.5 and by hypothesis, we have

$$\begin{aligned} f^{-1}(i_{\mu_Y^m}(B)) &= f^{-1}(Y\backslash c_{\mu_Y^m}(Y\backslash B)) \\ &= X\backslash f^{-1}(c_{\mu_Y^m}(Y\backslash B)) \\ &\subseteq X\backslash gbc_{\mu_X^n}(f^{-1}(c_{\mu_Y^m}(i_{\mu_Y^n}(c_{\mu_Y^m}(Y\backslash B)))) \\ &= X\backslash gbc_{\mu_X^n}(f^{-1}(c_{\mu_Y^m}(i_{\mu_Y^n}(Y\backslash i_{\mu_Y^m}(B))))) \\ &= X\backslash gbc_{\mu_X^n}(f^{-1}(c_{\mu_Y^m}(Y\backslash c_{\mu_Y^n}(i_{\mu_Y^m}(B))))) \\ &= X\backslash gbc_{\mu_X^n}(f^{-1}(Y\backslash i_{\mu_Y^m}(c_{\mu_Y^n}(i_{\mu_Y^m}(B))))) \\ &= X\backslash gbc_{\mu_X^n}(X\backslash f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(i_{\mu_Y^m}(B))))) \\ &= X\backslash(X\backslash gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(i_{\mu_Y^m}(B)))))) \\ &= gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(i_{\mu_Y^m}(B))))) \end{aligned}$$

(v)$\Rightarrow$ (vi): Let V be any $\mu^{(m,n)}$-regular open subset of Y. Then V is μ_Y^m-open in Y. Hence, $V = i_{\mu_Y^m}(V)$. Since V is $\mu^{(m,n)}$-regular open, $V = i_{\mu_Y^m}(c_{\mu_Y^n}(V)) = i_{\mu_Y^m}(c_{\mu_Y^n}(i_{\mu_Y^m}(V)))$. By (v),

$$\begin{aligned} f^{-1}(i_{\mu_Y^m}(V) &= f^{-1}(V) \\ &\subseteq gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(i_{\mu_Y^m}(V))))) \\ &= gbi_{\mu_X^n}(f^{-1}(V)) \\ &\subseteq f^{-1}(V). \end{aligned}$$

Therefore, $f^{-1}(V) = gbi_{\mu_X^n}(f^{-1}(V))$.

(vi)$\Rightarrow$ (vii): Let F be any $\mu^{(m,n)}$-regular closed subset of Y. Then $Y\backslash F$ is a $\mu^{(m,n)}$-regular open subset of Y. By (vi),

$$\begin{aligned} f^{-1}(Y\backslash F) &= gbi_{\mu_X^n}(f^{-1}(Y\backslash F)) \\ X\backslash f^{-1}(F) &= gbi_{\mu_X^n}(X\backslash f^{-1}(F)) \\ &= X\backslash gbc_{\mu_X^n}(f^{-1}(F)). \end{aligned}$$

Therefore, $f^{-1}(F) = gbc_{\mu_X^n}(f^{-1}(F))$.

(vii)$\Rightarrow$ (i): Let $x \in X$ and let V be any μ_Y^m-open set in Y with $f(x) \in V$. Then, $V = i_{\mu_Y^m}(V) \subseteq i_{\mu_Y^m}(c_{\mu_Y^n}(V))$. Since $i_{\mu_Y^m}(c_{\mu_Y^n}(V))$ is a $\mu^{(m,n)}$-regular open, by (vii), $f^{-1}(Y\backslash(i_{\mu_Y^m}(c_{\mu_Y^n}(V)))) = gbc_{\mu_X^n}(f^{-1}(Y\backslash(i_{\mu_Y^m}(c_{\mu_Y^n}(V)))))$. Thus,

$$\begin{aligned} X\backslash f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))) &= gbc_{\mu_X^n}(X\backslash f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V)))) \\ &= X\backslash gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V)))). \end{aligned}$$

It follows that $f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))) = gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))))$. Since $f(x) \in V \subseteq i_{\mu_Y^m}(c_{\mu_Y^n}(V))$, $x \in f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))) = gbi_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))))$. Hence, there exists a $g\mu_X^n b$-open set O with $x \in O \subseteq f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))$. This implies that $f(O) \subseteq i_{\mu_Y^m}(c_{\mu_Y^n}(V))$. Therefore, f is almost $g\mu^{(m,n)}b$-continuous. $\square$

Definition 3.17 A function $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ is said to be *regular strongly $g\mu^{(m,n)}b$-continuous* (briefly rs-$g\mu^{(m,n)}b$-continuous) if for every $g\mu_Y^m$-open set U in Y, $f^{-1}(U)$ is μ_X^n-open in X. It is *pairwise rs-$g\mu b$-continuous* if it is both rs-$g\mu^{(1,2)}b$-continuous and rs-$g\mu^{(2,1)}b$-continuous.

Lemma 3.18 [4] *Let (X, μ_X^1, μ_X^2) and (Y, μ_Y^1, μ_Y^2) be BGTS and $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ be a bijective function. Then the following are equivalent:*

1. *f is rs-$g\mu^{(m,n)}b$-continuous*
2. *For each $g\mu_Y^m b$-closed subset F of Y, $f^{-1}(F)$ is a μ_X^n-closed subset of X.*

Theorem 3.19 *A function $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ is rs-$g\mu^{(m,n)}b$-continuous if and only if for each $x \in X$ and for every $g\mu_Y^m b$-open set V containing $f(x)$, there exists a μ_X^n-open set U containing x such that $f(U) \subseteq V$.*

Proof: Let $x \in X$ and V be a $g\mu_Y^m b$-open set with $f(x) \in V$. Since f is rs-$g\mu^{(m,n)}b$-continuous, $f^{-1}(V)$ is μ_X^n-open in X. This implies that $x \in f^{-1}(V) = i_{\mu_X^n}(f^{-1}(V))$. Hence, there exists a μ_X^n-open set U with $x \in U$ and $U \subseteq f^{-1}(V)$. Thus, $f(U) \subseteq V$.

Conversely, let V be a $g\mu_Y^m b$-open set in Y. We claim that $f^{-1}(V)$ is μ_X^n-open in X. Suppose that $x \in f^{-1}(V)$. Then $f(x) \in V$. By assumption, there exists a μ_X^n-open set U with $x \in U$ and $f(U) \subseteq V$ that is $U \subseteq f^{-1}(V)$. Hence, $x \in i_{\mu_X^n}(f^{-1}(V))$. This implies that $f^{-1}(V) \subseteq i_{\mu_X^n}(f^{-1}(V)$. Thus $f^{-1}(V)$ is open. Therefore, f is an rs-$g\mu^{(m,n)}b$-continuous. $\square$

Theorem 3.20 *If* $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ *is an* rs-$g\mu^{(m,n)}b$*-continuous, then* f *is* $g\mu^{(m,n)}b$*-continuous.*

Proof: Let V be a μ_Y^m-open set in Y. Then V is a $g\mu_Y^m b$-open set in Y. Since f is rs-$g\mu^{(m,n)}b$-continuous, $f^{-1}(V)$ is μ_X^n-open in X. Hence, $f^{-1}(V)$ is $g\mu_X^n b$-open in X. Therefore, f is $g\mu^{(m,n)}b$-continuous. $\square$

Remark 3.21 *The converse of Theorem 3.20 is not true.*

To see this consider the BGTS (X, μ_X^1, μ_X^2) and (Y, μ_Y^1, μ_Y^2) in Example 3.3. Then all subsets of Y are $g\mu_Y^1 b$-open in Y. Define $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ by $f(a) = f(c) = d$, $f(b) = e$. Then f is $g\mu^{(1,2)}b$-continuous but not rs-$g\mu^{(1,2)}b$-continuous since $\{e\}$ is a $g\mu_Y^1 b$-open set in Y but $f^{-1}(\{e\}) = \{b\}$ is not a μ_X^2-open set in X.

As a consequence of Theorem 3.20, the properties stated in Theorem 3.8 hold for an rs-$g\mu^{(m,n)}b$-continuous function $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$.

Theorem 3.22 *A function* $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ *is* rs-$g\mu^{(m,n)}b$*- continuous if and only if* $f^{-1}(F)$ *is* μ_X^n*-closed for every* $g\mu_Y^m b$*-closed set* F *in* Y*.*

Proof: Suppose f is rs-$g\mu^{(m,n)}b$-continuous function. Let F be a $g\mu_Y^m b$-closed set in Y. Then $f^{-1}(Y \backslash F) = X \backslash f^{-1}(F)$ is μ_X^n-open in X. It follows that $f^{-1}(F)$ is μ_X^n-closed in X.

For the converse, let U be a $g\mu_Y^n b$-open set in Y. Then $Y \backslash U$ is a $g\mu_Y^m b$-closed set in Y. By assumption, $f^{-1}(Y \backslash U) = X \backslash f^{-1}(U)$ is μ_X^n-closed in X. Therefore, $f^{-1}(U)$ is μ_X^n-open in X. This implies that f is an rs-$g\mu^{(m,n)}b$-continuous function. $\square$

Theorem 3.23 *Let* $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ *is* rs-$g\mu^{(m,n)}b$*-continuous function. Then the following hold:*

(i) $f(gbc_{\mu_X^n}(A)) \subseteq gbc_{\mu_Y^m}(f(A))$ *for every* $A \subseteq X$*;*

(ii) $gbc_{\mu_X^n}(f^{-1}(B)) \subseteq f^{-1}(gbc_{\mu_Y^m}(B))$ *for every* $B \subseteq Y$*.*

Proof: Suppose that f is an rs-$g\mu^{(m,n)}b$-continuous function.
(i) Let $A \subseteq X$ and let $x \in gbc_{\mu_X^n}(A)$. Suppose that O is a $g\mu_Y^m b$-open set with $f(x) \in O$. Since f is rs-$g\mu^{(m,n)}b$-continuous, $f^{-1}(O)$ is μ_X^n-open. Thus, $f^{-1}(O)$ is a $g\mu_X^n b$-open and $x \in f^{-1}(O)$. Since $x \in gbc_{\mu_X^n}(A)$, $A \cap f^{-1}(O) \neq \varnothing$. This implies that $f(A) \cap O \neq \varnothing$. Therefore, $f(x) \in gbc_{\mu_Y^m}(f(A))$. Accordingly, $f(gbc_{\mu_X^n}(A)) \subseteq gbc_{\mu_Y^m}(f(A))$.
(ii) Let $B \subseteq Y$. By (i), $f(gbc_{\mu_X^n}(f^{-1}(B))) \subseteq gbc_{\mu_Y^m}(f(f^{-1}(B))) \subseteq gbc_{\mu_Y^m}(B)$. Therefore, $gbc_{\mu_X^n}(f^{-1}(B)) \subseteq f^{-1}(gbc_{\mu_Y^m}(B))$. $\square$

References

[1] Boonpok C., *Weakly open functions on bigeneralized topological spaces*, Int. Journal of Math. Analysis, Vol.6, 2012.

[2] Császár, Á., *Generalized Open Sets in Generalized Topologies*, Acta Mathematica Hungarica. 106(1-2). (2005), 53-66. http://dx.doi.org/10.1007/s10474-005-0005-5

[3] Császár, Á., *Generalized Topology, Generalized Continuity.* Acta Mathematica Hungarica **96** (2002), 351-357. http://dx.doi.org/10.1023/a:1019713018007

[4] Donesa, M., Rara, H., *Some gb-Separation Axioms in Bigeneralized Topological Spaces.* Applied Mathematical Sciences, Vol. 9, 2015, no. 22, 1051 - 1060. http://dx.doi.org/10.12988/ams.2015.411990

[5] Duangphui T., Boonpok C., Viriyapong C., *Continuous Functions on Bigeneralized Topological Spaces*, Int. Journal of Math. Analysis, Vol. 5, 2011, no. 24, 1165 - 1174.

[6] Min,W. K., *Almost continuity on generalized topological spaces*, Acta Math. Hungar., 125 (2009), 121-125. http://dx.doi.org/10.1007/s10474-009-8230-y

[7] Levine, N., *"Generalized Closed Sets in Topology"*. Rend. Circ. Mat. Palermo, 19(1982), pp 82-88.

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