Generalized $\mu^{(m,n)}b$-Continuous Functions in Bigeneralized Topological Spaces

Methos Kristy Villar Donesa$^1$ and Helen Moso Rara

Department of Mathematics and Statistics
Mindanao State University-Iligan Institute of Technology
Tibanga, Iligan city, Philippines

Abstract

The purpose of this paper is to introduce the notions of $g\mu^{(m,n)}b$-continuous, almost $g\mu^{(m,n)}b$-continuous and regular strongly $g\mu^{(m,n)}b$-continuous functions in bigeneralized topological spaces. Basic properties, characterizations and relationships between these functions are obtained.

Keywords: bigeneralized topological spaces, $g\mu^{(m,n)}b$-continuous, almost $g\mu^{(m,n)}b$-continuous and regular strongly $g\mu^{(m,n)}b$-continuous functions

1 Introduction

Császár introduced the concepts of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. In particular, he investigated characterizations for the generalized continuous function by using a closure operator defined on generalized neighborhood systems. In [6], W. K. Min introduced the notion of almost $(g,g')$-continuity and investigated properties of such functions and relationships among $(g,g')$-continuity, almost $(g,g')$-continuity and weak...
(g, g')-continuity. In 2010, C. Boonpok [1] introduced the concepts of bigeneralized topological spaces and studied (m, n)-closed sets and (m, n)-open sets in bigeneralized topological spaces. He also introduced the notion of weakly open functions in bigeneralized topological spaces and investigated properties of such functions. Recently, in 2011, T. Duangphui, C. Boonpok, and C. Viriyapong [5] introduced the notions of (µ, µ')(m, n)-continuous and almost (µ, µ')(m, n)-continuous functions. They obtained many characterizations and properties of such functions.

In this paper the concepts of generalized µ(m, n)b-continuous (briefly gµ(m, n)b-continuous), almost gµ(m, n)b-continuous and regular strongly gµ(m, n)b-continuous (briefly rs-gµ(m, n)b-continuous) functions are defined and characterized.

2 Preliminaries

Definition 2.1 [3] Let X be a nonempty set. A collection µ of subsets of X is a generalized topology (or briefly GT) on X if it satisfies the following:

(O1) ∅ ∈ µ; and

(O2) If {Mi : i ∈ I} ⊆ µ, then \( \bigcup_{i \in I} M_i \in \mu. \)

If µ is a GT on X, then (X, µ) is called a generalized topological space (or briefly GTS), and the elements of µ are called µ-open sets. A subset F of X is said to be µ-closed if the complement X \ F of F is µ-open.

Definition 2.2 [2] Let (X, µ) be a GTS and let A ⊆ X. The µ-interior of A, denoted by \( i_\mu(A) \), is the union of all µ-open sets contained in A. The µ-closure of A, denoted by \( c_\mu(A) \), is the intersection of all µ-closed sets containing A.

Definition 2.3 [2] Let (X, µ) be a GTS and A ⊆ X. Then

(i) A is said to be µb-open if \( A \subseteq c_\mu(i_\mu(A)) \cup i_\mu(c_\mu(A)) \). A set is µb-closed if its complement is µb-open.

(ii) The µb-closure of A, denoted by \( bc_\mu(A) \) is the intersection of all µb-closed sets containing A.

(iii) The µb-interior of A, denoted by \( bi_\mu(A) \) is the union of all µb-open sets contained in A.

Definition 2.4 [7] Let (X, µ) be a GTS and A ⊆ X. Then
(i) $A$ is said to be generalized $\mu b$-open (briefly $g\mu b$-open) if $U \subseteq A$ for some $\mu$-closed set $U$, then $U \subseteq bi_{\mu}(A)$. The complement of a $g\mu b$-open set is called a $g\mu b$-closed set. Equivalently, $A$ is $g\mu b$-closed if $A \subseteq U$ for some $\mu$-open set $U$, then $bc_{\mu}(A) \subseteq U$.

(ii) The $g\mu b$-closure of $A$, denoted by $gbc_{\mu}(A)$, is the intersection of all $g\mu b$-closed sets containing $A$. Thus, $A \subseteq gbc_{\mu}(A)$.

(iii) The $g\mu b$-interior of $A$, denoted by $gbi_{\mu}(A)$, is the union of all $g\mu b$-open sets contained in $A$. Hence, $gbi_{\mu}(A) \subseteq A$.

The proofs of the following are straightforward.

**Theorem 2.5** Let $(X, \mu)$ be a GTS and $A \subseteq X$.

(i) If $A$ is $\mu$-open, then $A$ is $g\mu b$-open.

(ii) $i_{\mu}(A) \subseteq bi_{\mu}(A)$.

(iii) If $A$ is $g\mu b$-open, then $gbi_{\mu}(A) = A$.

(iv) If $A$ is $g\mu b$-closed, then $gbc_{\mu}(A) = A$.

(v) $i_{\mu}(A) = X \setminus c_{\mu}(X \setminus A)$

(vi) $gbi_{\mu}(A) = X \setminus gbc_{\mu}(X \setminus A)$

**Theorem 2.6** [4] Let $(X, \mu)$ be a GTS and $A \subseteq X$. Then $q \in gbc_{\mu}(A)$ if and only if for every $g\mu b$-open set $O$ with $q \in O$, $O \cap A \neq \emptyset$.

**Definition 2.7** [1] Let $X$ be a nonempty set and let $\mu_1, \mu_2$ be generalized topologies on $X$. The triple $(X, \mu_1, \mu_2)$ is said to be a bigeneralized topological space (briefly $BGTS$).

Throughout this paper, $m$ and $n$ are elements of the set $\{1, 2\}$ where $m \neq n$. Let $(X, \mu_1, \mu_2)$ be a $BGTS$ and $A$ a subset of $X$. The closure of $A$ and the interior of $A$ with respect to $\mu_m$ are denoted by $c_{\mu_m}(A)$ and $i_{\mu_m}(A)$, respectively. The $gb$-closure of $A$ and the $gb$-interior of $A$ with respect to $\mu_m$ are denoted by $gbc_{\mu_m}(A)$ and $gbi_{\mu_m}(A)$, respectively.
3 $g\mu^{(m,n)}b$-Continuous functions

In this section, we introduce $g\mu^{(m,n)}b$-continuous, almost $g\mu^{(m,n)}b$-continuous and rs-$g\mu^{(m,n)}b$-continuous functions in bigeneralized topological spaces and study some of their properties and relationships..

All throughout this paper $(X,\mu_X^1,\mu_X^2)$ and $(Y,\mu_Y^1,\mu_Y^2)$ are bigeneralized topological spaces.

**Definition 3.1** A function $f : (X,\mu_X^1,\mu_X^2) \to (Y,\mu_Y^1,\mu_Y^2)$ is said to be $g\mu^{(m,n)}b$-continuous if for each $\mu_Y^m$-open set $V$, $f^{-1}(V)$ is $g\mu_X^n b$-open set in $X$.

**Definition 3.2** A function $f : (X,\mu_X^1,\mu_X^2) \to (Y,\mu_Y^1,\mu_Y^2)$ is said to be pairwise $g\mu b$-continuous if $f$ is both $g\mu^{(1,2)}b$-continuous and $g\mu^{(2,1)} b$-continuous.

**Example 3.3** Let $X = \{a,b,c\}$ and $Y = \{d,e\}$. Consider the generalized topologies $\mu_X^1 = \{\emptyset,\{a\},\{b,c\},X\}$, $\mu_X^2 = \{\emptyset,\{e\},\{b,c\}\}$, $\mu_Y^1 = \{\emptyset,\{d\}\}$, and $\mu_Y^2 = \{\emptyset,\{e\},Y\}$. All subsets of $X$ are $g\mu_X^1 b$-open and the $g\mu_X^2 b$-open sets are $\emptyset,\{a\},\{b\},\{c\},\{b,c\},\{a,c\}$ and $X$. Let $f : (X,\mu_X^1,\mu_X^2) \to (Y,\mu_Y^1,\mu_Y^2)$ be a function defined by $f(a) = f(c) = d$, $f(b) = e$. Then $f$ is $g\mu^{(1,2)}b$-continuous and $g\mu^{(2,1)} b$-continuous. Thus, $f$ is pairwise $g\mu b$-continuous.

**Theorem 3.4** For a function $f : (X,\mu_X^1,\mu_X^2) \to (Y,\mu_Y^1,\mu_Y^2)$, the following properties are equivalent.

(i) For every $x \in X$ and for each $\mu_Y^m$-open set $V$ in $Y$ containing $f(x)$, there exists a $g\mu_X^n b$-open set $U$ containing $x$ such that $f(U) \subseteq V$;

(ii) $x \in gbi_{\mu_X^2}(f^{-1}(V))$ for every $V \in \mu_Y^m$ containing $f(x)$;

(iii) $x \in gbi_{\mu_X^2}(f^{-1}(B))$ for every subset $B$ of $Y$ with $x \in f^{-1}(i_{\mu_Y^m}(B))$;

(iv) $x \in f^{-1}(F)$ for every $\mu_Y^m$-closed subset $F$ of $Y$ such that $x \in gbc_{\mu_X^2}(f^{-1}(F))$.

**Proof:** Let $f : (X,\mu_X^1,\mu_X^2) \to (Y,\mu_Y^1,\mu_Y^2)$ be a function and $x \in X$.

(i) $\Rightarrow$ (ii): Let $V$ be any $\mu_Y^m$-open subset of $Y$ such that $f(x) \in V$. By (1), there exists a $g\mu_X^n b$-open set $U \subseteq X$ with $x \in U$ and $f(U) \subseteq V$. Hence, $U \subseteq f^{-1}(V)$ with $x \in U$. This implies that $x \in gbi_{\mu_X^2}(f^{-1}(V))$.

(ii) $\Rightarrow$ (i): Let $V \in \mu_Y^m$ with $f(x) \in V$. By (ii), $x \in gbi_{\mu_X^2}(f^{-1}(V))$. Hence, there exists a $g\mu_X^n b$-open set $U \subseteq f^{-1}(V)$ such that $x \in U$.

(ii) $\Rightarrow$ (iii): Let $B \subseteq Y$ with $x \in f^{-1}(i_{\mu_Y^m}(B))$. Then $f(x) \in i_{\mu_Y^m}(B)$. Since $i_{\mu_Y^m}(B) \in \mu_Y^m$, by (ii), $x \in gbi_{\mu_X^2}(f^{-1}(i_{\mu_Y^m}(B))) \subseteq gbi_{\mu_X^2}(f^{-1}(B))$. Thus, $x \in gbi_{\mu_X^2}(f^{-1}(B))$.

(iii) $\Rightarrow$ (iv): Let $F$ be a $\mu_Y^m$-closed subset of $Y$ such that $x \notin f^{-1}(F)$. Then $x \in X \setminus f^{-1}(F) = f^{-1}(Y \setminus F) = f^{-1}(i_{\mu_Y^m}(Y \setminus F))$ since $Y \setminus F$ is $\mu_Y^m$-open. By (iii),
Lemma 3.7 are equivalent:

(iv) \[ x \in gbi_{\mu_X} (f^{-1}(Y \setminus F)) = gbi_{\mu_X} (X \setminus f^{-1}(F)) = X \setminus gbc_{\mu_X} (f^{-1}(F)). \]

Then for every \( x \in X \), there exist a \( \mu \)-open set \( V \) containing \( x \) such that \( f(V) \subseteq V \).

The converse of Theorem 3.5 is not true.

Proof: Let \( x \in X \) and let \( V \) be any \( \mu \)-open set in \( Y \) with \( f(x) \in V \). Since \( f \) is \( \mu \)-continuous, \( f^{-1}(V) \) is \( \mu \)-open in \( X \). Take \( U = f^{-1}(V) \). Then there exist a \( \mu \)-open set \( U \) with \( x \in U \) and \( f(U) \subseteq V \).

Remark 3.6 The converse of Theorem 3.5 is not true.

To see this consider the BGTS spaces \((X, \mu_X, \mu_Y)\) and \((Y, \mu_Y, \mu_Y)\) in Example 3.3. Define \( f : (X, \mu_X, \mu_Y) \rightarrow (Y, \mu_Y, \mu_Y)\) by \( f(a) = f(b) = d, f(c) = e \). Then for \( a \in X \) where \( f(a) \in \{d\} \in \mu_Y \), there exists a \( \mu \)-open set \( \{a\} \) and \( f(\{a\}) = \{d\} \). But \( f \) is not \( \mu \)-continuous.

Lemma 3.7 Let \((X, \mu_X, \mu_Y)\) and \((Y, \mu_Y, \mu_Y)\) be BGTS. Then the following are equivalent:

(i) \( f^{-1}(V) = gbi_{\mu_Y} (f^{-1}(V)) \) for every \( V \in \mu_Y \);

(ii) \( f^{-1}(i_{\mu_Y}(B)) \subseteq gbi_{\mu_Y} (f^{-1}(B)) \) for every \( B \subseteq Y \);

(iii) \( gbc_{\mu_Y} (f^{-1}(F)) = f^{-1}(F) \) for every \( \mu_Y \)-closed subset \( F \) of \( Y \).

Proof: (i) \( \Rightarrow \) (ii): Let \( B \subseteq Y \). Since \( i_{\mu_Y}(B) \in \mu_Y \), by (i), \( f^{-1}(i_{\mu_Y}(B)) = gbi_{\mu_Y} (f^{-1}(i_{\mu_Y}(B))) \subseteq gbi_{\mu_Y} (f^{-1}(B)). \)

(ii) \( \Rightarrow \) (iii): Let \( F \) be a \( \mu \)-closed subset of \( Y \) such that \( x \notin f^{-1}(F) \). Then \( x \in X \setminus f^{-1}(F) = f^{-1}(Y \setminus F) = f^{-1}(i_{\mu_Y}(Y \setminus F)) \). By (ii), \( x \in gbi_{\mu_Y} (f^{-1}(Y \setminus F)) = gbi_{\mu_Y} (X \setminus f^{-1}(F)) = X \setminus gbc_{\mu_Y} (f^{-1}(F)). \) Hence, \( x \notin gbc_{\mu_Y} (f^{-1}(F)) \). Therefore, \( gbc_{\mu_Y} (f^{-1}(F)) \subseteq f^{-1}(F) \subseteq gbc_{\mu_Y} (f^{-1}(F)). \) It follows that \( f^{-1}(F) = gbc_{\mu_Y} (f^{-1}(F)). \)

(iii) \( \Rightarrow \) (i): Let \( V \in \mu_Y \). Then \( Y \setminus V \) is a \( \mu \)-closed set in \( Y \). By (iii), \( gbc_{\mu_Y} (f^{-1}(Y \setminus V)) = f^{-1}(Y \setminus V) = X \setminus f^{-1}(V). \) Thus, \( X \setminus gbi_{\mu_Y} (f^{-1}(V)) = X \setminus f^{-1}(V). \) Hence, \( f^{-1}(V) = gbi_{\mu_Y} (f^{-1}(V)). \)

Theorem 3.8 Let \((X, \mu_X, \mu_Y)\) and \((Y, \mu_Y, \mu_Y)\) be BGTS. For a \( \mu \)-continuous function \( f : (X, \mu_X, \mu_X) \rightarrow (Y, \mu_Y, \mu_Y) \), the following properties hold.
Theorem 3.15

(i) \( f^{-1}(V) = gbi_{\mu_X}^{-1}(f^{-1}(V)) \) for every \( V \in \mu_Y^m \);
(ii) \( f^{-1}(i_{\mu_{Y}}(B)) \subseteq gbi_{\mu_X}^{-1}(f^{-1}(B)) \) for every \( B \subseteq Y \);
(iii) \( gbi_{\mu_X}^{-1}(f^{-1}(F)) = f^{-1}(F) \) for every \( \mu_Y^m \)-closed subset \( F \) of \( Y \).

Proof: Let \( f : (X, \mu_X^1, \mu_X^2) \rightarrow (Y, \mu_Y^1, \mu_Y^2) \) be a \( g\mu^{(m,n)}_{\mu_X^2} \)-continuous function.

(i): Let \( V \in \mu_Y^m \). Since \( f \) is \( g\mu^{(m,n)}_{\mu_X^2} \)-continuous \( f^{-1}(V) \) is \( \mu_Y^m \)-open in \( X \). Hence, \( f^{-1}(V) = gbi_{\mu_X}^{-1}(f^{-1}(V)) \).

(ii) and (iii): Follow immediately from Lemma 3.7. \( \square \)

Remark 3.9 As a consequence of Theorems 3.4, 3.5, and Lemma 3.7, the converse of Theorem 3.8 is not true.

Definition 3.10 A function \( f : (X, \mu_X^1, \mu_X^2) \rightarrow (Y, \mu_Y^1, \mu_Y^2) \) is said to be almost \( g\mu^{(m,n)}_{\mu_X^2} \)-continuous at \( x \in X \) if for each \( \mu_Y^m \)-open set \( V \) containing \( f(x) \), there exists a \( \mu_X^m \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq i_{\mu_Y^m}(c_{\mu_Y^m}(V)) \).

Definition 3.11 A function \( f : (X, \mu_X^1, \mu_X^2) \rightarrow (Y, \mu_Y^1, \mu_Y^2) \) is said to be almost \( g\mu^{(m,n)}_{\mu_X^2} \)-continuous if \( f \) is almost \( g\mu^{(m,n)}_{\mu_X^2} \)-continuous at every \( x \in X \). It is pairwise almost \( g\mu_{\mu_X^2} \)-continuous if it is almost \( g\mu^{(1,2)}_{\mu_X^2} \)-continuous and almost \( g\mu^{(2,1)}_{\mu_X^2} \)-continuous.

Theorem 3.12 If \( f : (X, \mu_X^1, \mu_X^2) \rightarrow (Y, \mu_Y^1, \mu_Y^2) \) is \( g\mu^{(m,n)}_{\mu_X^2} \)-continuous, then \( f \) is almost \( g\mu^{(m,n)}_{\mu_X^2} \)-continuous.

Proof: Let \( x \in X \) and \( V \) be any \( \mu_Y^m \)-open set in \( Y \) with \( f(x) \in V \). Then \( f^{-1}(V) \) is \( g\mu_X^1 \)-open in \( X \) and \( x \in f^{-1}(V) \). Let \( U = f^{-1}(V) \). Then \( f(U) \subseteq V \subseteq i_{\mu_Y^m}(c_{\mu_Y^m}(V)) \). Therefore, \( f \) is almost \( g\mu^{(m,n)}_{\mu_X^2} \)-continuous. \( \square \)

Remark 3.13 The converse of Theorem 3.12 is not true.

To see this let \( X = \{a, b, c\} = Y \), \( \mu_X^1 = \{\emptyset, \{a\}, \{b, c\}, X\} \), \( \mu_X^2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( \mu_Y^1 = \{\emptyset, \{a\}, \{b\}, \{a\}\} \). Then all subsets of \( X \) are \( g\mu_X^1 \)-open sets and \( g\mu_X^2 \)-open sets of \( X \) are \( \emptyset, \{b\}, \{a, b\}, \{a\}\) and \( X \). Let \( f : (X, \mu_X^1, \mu_X^2) \rightarrow (Y, \mu_Y^1, \mu_Y^2) \) be the identity function. Then \( f \) is almost \( g\mu^{(1,2)}_{\mu_X^2} \)-continuous but it is not \( g\mu^{(1,2)}_{\mu_X^2} \)-continuous.

Definition 3.14 Let \( (X, \mu_X^1, \mu_X^2) \) be a BGTS. Then \( A \subseteq X \) is \( \mu^{(m,n)}_{\mu_X^2} \)-regular open if \( A = i_{\mu_Y^m}(c_{\mu_Y^m}(A)) \).

Theorem 3.15 For a function \( f : (X, \mu_X^1, \mu_X^2) \rightarrow (Y, \mu_Y^1, \mu_Y^2) \), the following properties are equivalent:

(i) \( f \) is almost \( g\mu^{(m,n)}_{\mu_X^2} \)-continuous at \( x \in X \);
(ii) \( x \in gbi_{\mu_X^m}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^m}(V)))) \) for every \( V \in \mu_Y^m \) containing \( f(x) \);

(iii) \( x \in gbi_{\mu_X^m}(f^{-1}(V)) \) for every \( \mu^{(m,n)} \)-regular open subset \( V \) of \( Y \) containing \( f(x) \);

(iv) For every \( \mu^{(m,n)} \)-regular open set \( V \) containing \( f(x) \), there exists a \( \mu_X^m \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).

Proof: (i) \( \Rightarrow \) (ii): Let \( V \in \mu_Y^m \) with \( f(x) \in V \). Then \( x \in f^{-1}(V) \). Since \( f \) is almost \( g\mu^{(m,n)} \)-continuous at \( x \), there exists a \( g\mu_X^m \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq i_{\mu_Y^m}(c_{\mu_Y^m}(V)) \). Thus, \( x \in U \subseteq f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^m}(V))) \). Since \( U \) is a \( g\mu_X^m \)-open set, we have \( x \in gbi_{\mu_X^m}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^m}(V)))) \).

(ii) \( \Rightarrow \) (iii): Let \( V \) be any \( \mu^{(m,n)} \)-regular open subset of \( Y \) containing \( f(x) \). Then \( f(x) \in V = i_{\mu_Y^m}(c_{\mu_Y^m}(V)) \). Since \( V \in \mu_Y^m \), by (ii),

\[
x \in gbi_{\mu_X^m}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^m}(V)))) = gbi_{\mu_X^m}(f^{-1}(V)).
\]

Hence, \( x \in gbi_{\mu_X^m}(f^{-1}(V)) \).

(iii) \( \Rightarrow \) (iv): Let \( V \) be any \( \mu^{(m,n)} \)-regular open subset of \( Y \) containing \( f(x) \). Then by (iii) \( x \in gbi_{\mu_X^m}(f^{-1}(V)) \). Hence, there exists a \( g\mu_X^m \)-open set \( U \) with \( x \in U \subseteq f^{-1}(V) \). Thus, \( f(U) \subseteq V \).

(iv) \( \Rightarrow \) (i): Let \( x \in X \) and \( V \) be a \( \mu_Y^m \)-open set in \( Y \) containing \( f(x) \). Then \( f(x) \in V = i_{\mu_Y^m}(V) \subseteq i_{\mu_Y^m}(c_{\mu_Y^m}(V)) \). Since \( i_{\mu_Y^m}(c_{\mu_Y^m}(V)) \) is \( \mu^{(m,n)} \)-regular open, by (iv), there exists a \( g\mu_X^m \)-open set \( U \) with \( x \in U \) and \( f(U) \subseteq i_{\mu_Y^m}(c_{\mu_Y^m}(V)) \), Therefore, \( f \) is almost \( g\mu^{(m,n)} \)-continuous at \( x \) in \( X \). \( \square \)

Theorem 3.16 For a function \( f : (X, \mu_X^1, \mu_X^2) \rightarrow (Y, \mu_Y^1, \mu_Y^2) \), the following properties are equivalent:

(i) \( f \) is almost \( g\mu^{(m,n)} \)-continuous;

(ii) \( f^{-1}(V) \subseteq gbi_{\mu_X^m}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^m}(V)))) \) for every \( V \in \mu_Y^m \);

(iii) \( gbc_{\mu_X^m}(f^{-1}(c_{\mu_Y^m}(i_{\mu_Y^m}(F)))) \subseteq f^{-1}(F) \) for every \( \mu_Y^m \)-closed set \( F \) in \( Y \);

(iv) \( gbc_{\mu_X^m}(f^{-1}(c_{\mu_Y^m}(i_{\mu_Y^m}(c_{\mu_Y^m}(B)))))) \subseteq f^{-1}(c_{\mu_Y^m}(B)) \) for every \( B \subseteq Y \);

(v) \( f^{-1}(i_{\mu_Y^m}(B)) \subseteq gbi_{\mu_X^m}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^m}(i_{\mu_Y^m}(B)))))) \) for every \( B \subseteq Y \);

(vi) \( f^{-1}(V) = gbi_{\mu_X^m}(f^{-1}(V)) \) for every \( \mu^{(m,n)} \)-regular open subset \( V \) of \( Y \);

(vii) \( f^{-1}(F) = gbc_{\mu_X^m}(f^{-1}(F)) \) for every \( \mu^{(m,n)} \)-regular closed subset \( F \) of \( Y \).
Proof: (i)⇒ (ii): Let \( V \in \mu_Y^m \) and \( x \in f^{-1}(V) \). Then \( f(x) \in V \). Since \( f \) is almost \( g\mu^{(m,n)} \)-continuous, there exists a \( g\mu^{n}_X \)-open set \( U \) with \( x \in U \) and \( f(U) \subseteq i_{\nu^{n}_V}(c_{\nu^{n}_V}(V)) \). Hence, \( x \in U \subseteq f^{-1}(i_{\nu^{n}_V}(c_{\nu^{n}_V}(V))) \). Thus, \( x \in gbi_{\mu^{n}_X}(f^{-1}(i_{\nu^{n}_V}(c_{\nu^{n}_V}(V)))) \). Therefore, \( f^{-1}(V) \subseteq gbi_{\mu^{n}_X}(f^{-1}(i_{\nu^{n}_V}(c_{\nu^{n}_V}(V)))) \).

(ii)⇒ (iii): Let \( F \) be any \( \mu^{n}_Y \)-closed set in \( Y \). Then \( Y \setminus F \) is a \( \mu^{n}_Y \)-open set in \( Y \). By (ii),

\[
X \setminus f^{-1}(F) = f^{-1}(Y \setminus F) \subseteq gbi_{\mu^{n}_X}(f^{-1}(i_{\nu^{n}_V}(c_{\nu^{n}_V}(Y \setminus F))))
\]

\[
\begin{align*}
&= gbi_{\mu^{n}_X}(f^{-1}(i_{\nu^{n}_V}(Y \setminus i_{\mu^{n}_V}(F)))) \\
&= gbi_{\mu^{n}_X}(f^{-1}(Y \setminus c_{\nu^{n}_V}(i_{\mu^{n}_V}(F)))) \\
&= gbi_{\mu^{n}_X}(X \setminus f^{-1}(c_{\nu^{n}_V}(i_{\mu^{n}_V}(F)))) \\
&= X \setminus gbc_{\mu^{n}_X}(f^{-1}(c_{\nu^{n}_V}(i_{\mu^{n}_V}(F))))
\end{align*}
\]

Hence, \( gbc_{\mu^{n}_X}(f^{-1}(c_{\nu^{n}_V}(i_{\mu^{n}_V}(F)))) \subseteq f^{-1}(F) \).

(iii)⇒ (iv): Let \( B \subseteq Y \). Since \( c_{\nu^{n}_V}(B) \) is \( \mu^{n}_Y \)-closed in \( Y \), by (iii),

\[
gbc_{\mu^{n}_X}(f^{-1}(c_{\nu^{n}_V}(i_{\mu^{n}_V}(c_{\nu^{n}_V}(B)))) \subseteq f^{-1}(c_{\nu^{n}_V}(B)).
\]

(iv)⇒ (v): Let \( B \subseteq Y \). By Theorem 2.5 and by hypothesis, we have

\[
f^{-1}(i_{\mu^{n}_V}(B)) = f^{-1}(Y \setminus c_{\nu^{n}_V}(Y \setminus B)) \\
= X \setminus f^{-1}(c_{\nu^{n}_V}(Y \setminus B)) \\
\subseteq X \setminus gbc_{\mu^{n}_X}(f^{-1}(c_{\nu^{n}_V}(i_{\mu^{n}_V}(c_{\nu^{n}_V}(Y \setminus B)))) \\
= X \setminus gbc_{\mu^{n}_X}(f^{-1}(c_{\nu^{n}_V}(i_{\mu^{n}_V}(Y \setminus i_{\mu^{n}_V}(B)))) \\
= X \setminus gbc_{\mu^{n}_X}(f^{-1}(Y \setminus i_{\mu^{n}_V}(c_{\nu^{n}_V}(i_{\mu^{n}_V}(B)))) \\
= X \setminus gbc_{\mu^{n}_X}(X \setminus f^{-1}(i_{\mu^{n}_V}(c_{\nu^{n}_V}(i_{\mu^{n}_V}(B)))) \\
= X \setminus (X \setminus gbc_{\mu^{n}_X}(f^{-1}(i_{\mu^{n}_V}(c_{\nu^{n}_V}(i_{\mu^{n}_V}(B)))) \\
= gbi_{\mu^{n}_X}(f^{-1}(i_{\mu^{n}_V}(c_{\nu^{n}_V}(i_{\mu^{n}_V}(B))))
\]

(v)⇒ (vi): Let \( V \) be any \( \mu^{(m,n)} \)-regular open subset of \( Y \). Then \( V \) is \( \mu^{m}_Y \)-open in \( Y \). Hence, \( V = i_{\nu^{n}_V}(V) \). Since \( V \) is \( \mu^{(m,n)} \)-regular open, \( V = i_{\nu^{n}_V}(c_{\nu^{n}_V}(V)) = i_{\nu^{n}_V}(c_{\nu^{n}_V}(i_{\mu^{n}_V}(V))) \). By (v),

\[
f^{-1}(i_{\mu^{n}_V}(V)) = f^{-1}(V) \\
\subseteq gbi_{\mu^{n}_X}(f^{-1}(i_{\mu^{n}_V}(c_{\nu^{n}_V}(i_{\mu^{n}_V}(V)))) \\
= gbi_{\mu^{n}_X}(f^{-1}(V)) \\
\subseteq f^{-1}(V).
\]

Therefore, \( f^{-1}(V) = gbi_{\mu^{n}_X}(f^{-1}(V)) \).
(vi)⇒ (vii): Let $F$ be any $\mu^{(m,n)}$-regular closed subset of $Y$. Then $Y \setminus F$ is a $\mu^{(m,n)}$-regular open subset of $Y$. By (vi),

$$f^{-1}(Y \setminus F) = gbi_{\mu_X}^{(m)}(f^{-1}(Y \setminus F))$$

$$X \setminus f^{-1}(F) = gbi_{\mu_X}^{(m)}(X \setminus f^{-1}(F)) = X \setminus gbi^{(m)}_{\mu_X}(f^{-1}(F)).$$

Therefore, $f^{-1}(F) = gbi_{\mu_X}^{(m)}(f^{-1}(F))$.

(vii)⇒ (i): Let $x \in X$ and let $V$ be any $\mu_X^{mn}$-open set in $Y$ with $f(x) \in V$. Then, $V = i_{\mu_Y}^{mn}(V) \subseteq i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V))$. Since $i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V))$ is a $\mu^{(m,n)}$-regular open, by (vii),

$$f^{-1}(Y \setminus (i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V)))) = gbi_{\mu_X}^{(m)}(f^{-1}(Y \setminus i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V))))).$$

Thus,

$$X \setminus f^{-1}(i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V))) = gbi_{\mu_X}^{(m)}(X \setminus f^{-1}(i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V)))) = X \setminus gbi_{\mu_X}^{(m)}(f^{-1}(i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V))))).$$

It follows that $f^{-1}(i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V))) = gbi_{\mu_X}^{(m)}(f^{-1}(i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V))))).$. Since $f(x) \in V \subseteq i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V))$, $x \in f^{-1}(i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V))) = gbi_{\mu_X}^{(m)}(f^{-1}(i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V))))).$. Hence, there exists a $\mu_X^{mn}$-$b$-open set $O$ with $x \in O \subseteq f^{-1}(i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V))).$ This implies that $f(O) \subseteq i_{\mu_Y}^{mn}(c_{\mu_Y}^{mn}(V))).$. Therefore, $f$ is almost $\mu^{(m,n)}_b$-continuous. □

**Definition 3.17** A function $f : (X, \mu_X, \mu_X') \to (Y, \mu_Y, \mu_Y')$ is said to be regular strongly $\mu^{(m,n)}_b$-continuous (briefly rs-$\mu^{(m,n)}_b$-continuous) if for every $\mu^{mn}_Y$-$open$ set $U$ in $Y$, $f^{-1}(U)$ is $\mu_X^{mn}$-open in $X$. It is pairwise rs-$\mu^{(m,n)}_b$-continuous if it is both rs-$\mu^{(1,2)}_b$-continuous and rs-$\mu^{(2,1)}_b$-continuous.

**Lemma 3.18** [4] Let $(X, \mu_X, \mu_X')$ and $(Y, \mu_Y, \mu_Y')$ be BGTS and $f : (X, \mu_X, \mu_X') \to (Y, \mu_Y, \mu_Y')$ be a bijective function. Then the following are equivalent:

1. $f$ is rs-$\mu^{(m,n)}_b$-continuous
2. For each $\mu^{mn}_X$-$closed$ subset $F$ of $Y$, $f^{-1}(F)$ is a $\mu_X^{mn}$-$closed$ subset of $X$.

**Theorem 3.19** A function $f : (X, \mu_X, \mu_X') \to (Y, \mu_Y, \mu_Y')$ is rs-$\mu^{(m,n)}_b$-continuous if and only if for each $x \in X$ and for every $\mu^{mn}_X$-$open$ set $V$ containing $f(x)$, there exists a $\mu_X^{mn}$-$open$ set $U$ containing $x$ such that $f(U) \subseteq V$.

**Proof:** Let $x \in X$ and $V$ be a $\mu^{mn}_X$-$open$ set with $f(x) \in V$. Since $f$ is rs-$\mu^{(m,n)}_b$-continuous, $f^{-1}(V)$ is $\mu_X^{mn}$-open in $X$. This implies that $x \in f^{-1}(V) = i_{\mu_X}^{mn}(f^{-1}(V))$. Hence, there exists a $\mu_X^{mn}$-$open$ set $U$ with $x \in U$ and $U \subseteq f^{-1}(V)$. Thus, $f(U) \subseteq V$.

Conversely, let $V$ be a $\mu^{mn}_X$-$open$ set in $Y$. We claim that $f^{-1}(V)$ is $\mu_X^{mn}$-open in $X$. Suppose that $x \in f^{-1}(V)$. Then $f(x) \in V$. By assumption, there exists a $\mu_X^{mn}$-$open$ set $U$ with $x \in U$ and $f(U) \subseteq V$ that is $U \subseteq f^{-1}(V)$. Hence, $x \in i_{\mu_X}^{mn}(f^{-1}(V))$. This implies that $f^{-1}(V) \subseteq i_{\mu_X}^{mn}(f^{-1}(V))$. Thus $f^{-1}(V)$ is open. Therefore, $f$ is an rs-$\mu^{(m,n)}_b$-continuous. □
**Theorem 3.20** If \( f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2) \) is an \( rs-g\mu^{(m,n)} \)-continuous function, then \( f \) is \( g\mu^{(m,n)} \)-continuous.

**Proof:** Let \( V \) be a \( \mu_Y^m \)-open set in \( Y \). Then \( V \) is a \( g\mu_Y^n \)-open set in \( Y \). Since \( f \) is \( rs-g\mu^{(m,n)} \)-continuous, \( f^{-1}(V) \) is \( \mu_X^1 \)-open in \( X \). Hence, \( f^{-1}(V) \) is \( \mu_X^1 \)-open set in \( X \). Therefore, \( f \) is \( g\mu^{(m,n)} \)-continuous.

**Remark 3.21** The converse of Theorem 3.20 is not true.

To see this consider the BGTS \((X, \mu_X^1, \mu_X^2) \) and \((Y, \mu_Y^1, \mu_Y^2) \) in Example 3.3. Then all subsets of \( Y \) are \( g\mu_Y^n \)-open in \( Y \). Define \( f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2) \) by \( f(a) = f(c) = d, f(b) = e \). Then \( f \) is \( \mu_Y^{(1,2)} \)-continuous but not \( rs-g\mu^{(1,2)} \)-continuous since \( \{e\} \) is a \( g\mu_Y^1 \)-open set in \( Y \) but \( f^{-1}(\{e\}) = \{b\} \) is not a \( \mu_X^1 \)-open set in \( X \).

As a consequence of Theorem 3.20, the properties stated in Theorem 3.8 hold for an \( rs-g\mu^{(m,n)} \)-continuous function \( f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2) \).

**Theorem 3.22** A function \( f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2) \) is \( rs-g\mu^{(m,n)} \)-continuous if and only if \( f^{-1}(F) \) is \( \mu_X^1 \)-open for every \( g\mu_Y^n \)-closed set \( F \) in \( Y \).

**Proof:** Suppose \( f \) is \( rs-g\mu^{(m,n)} \)-continuous. Let \( F \) be a \( g\mu_Y^n \)-closed set in \( Y \). Then \( f^{-1}(F) \) is \( \mu_X^1 \)-open in \( X \). It follows that \( f^{-1}(F) \) is \( \mu_X^1 \)-closed in \( X \).

For the converse, let \( U \) be a \( g\mu_Y^n \)-open set in \( Y \). Then \( Y \setminus U \) is a \( g\mu_Y^n \)-closed set in \( Y \). By assumption, \( f^{-1}(Y \setminus U) = X \setminus f^{-1}(U) \) is \( \mu_X^1 \)-closed in \( X \). Therefore, \( f^{-1}(U) \) is \( \mu_X^1 \)-open in \( X \). This implies that \( f \) is an \( rs-g\mu^{(m,n)} \)-continuous function.

**Theorem 3.23** Let \( f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2) \) be \( rs-g\mu^{(m,n)} \)-continuous function. Then the following hold:

(i) \( f(\text{gbc}_{\mu_X}^1(A)) \subseteq \text{gbc}_{\mu_Y}^1(f(A)) \) for every \( A \subseteq X \);

(ii) \( \text{gbc}_{\mu_X}^2(f^{-1}(B)) \subseteq f^{-1}(\text{gbc}_{\mu_Y}^2(B)) \) for every \( B \subseteq Y \).

**Proof:** Suppose that \( f \) is an \( rs-g\mu^{(m,n)} \)-continuous function.

(i) Let \( A \subseteq X \) and let \( x \in \text{gbc}_{\mu_X}^1(A) \). Suppose that \( O \) is a \( g\mu_Y^n \)-open set with \( f(x) \in O \). Since \( f \) is \( rs-g\mu^{(m,n)} \)-continuous, \( f^{-1}(O) \) is \( \mu_X^1 \)-open. Thus, \( f^{-1}(O) \) is a \( g\mu_X^1 \)-open and \( x \in f^{-1}(O) \). Since \( x \in \text{gbc}_{\mu_X}^1(A) \), \( A \cap f^{-1}(O) \neq \emptyset \). This implies that \( f(A) \cap O \neq \emptyset \). Therefore, \( f(x) \in \text{gbc}_{\mu_Y}^1(f(A)) \). Accordingly, \( f(\text{gbc}_{\mu_X}^1(A)) \subseteq \text{gbc}_{\mu_Y}^1(f(A)) \).

(ii) Let \( B \subseteq Y \). By (i), \( f(\text{gbc}_{\mu_X}^2(f^{-1}(B))) \subseteq \text{gbc}_{\mu_Y}^2(f(f^{-1}(B))) \subseteq \text{gbc}_{\mu_Y}^2(B) \). Therefore, \( \text{gbc}_{\mu_X}^2(f^{-1}(B)) \subseteq f^{-1}(\text{gbc}_{\mu_Y}^2(B)) \).
References


Received: February 13, 2015; Published: March 21, 2015