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Generalized Simpson-like Type Integral Inequalities for Differentiable Convex Functions via Riemann-Liouville Integrals

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Abstract

In this paper, by setting up a generalized integral identity for differentiable functions via Riemann-Liouville fractional integrals, the author derive new estimates on generalization of Simpson-like types inequalities for functions whose derivatives in the absolute value at certain powers are convex and s-convex in the second sense.

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1 Introduction

Let $f : I \subseteq R_+ \to R$ be a function defined on the interval I of real numbers. Then f is called to be s-convex in the second sense on I if the following inequality

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in I$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. This class of s-convex functions in the second sense is usually denoted by K_s^2.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

There are many results associated with convex functions in the area of inequalities, but some of those is the classical Hermite-Hadamard and Simpson type inequality, respectively [11, 14]:

Theorem 1.1. *Let $f : I \subseteq R \to R$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the following double inequality holds:*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Theorem 1.2. [12] *Let $f : I \subseteq R \to R$ be a differentiable function on the interior I^0 of an interval I and $a, b \in I^0$ with $a < b$. If $| f'(x) | \leq M, x \in [a, b]$, then the following inequality holds:*

$$\Big|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\Big| \leq \frac{M}{b-a}\Big[\frac{(x-a)^2+(b-x)^2}{2}\Big]. \tag{2}$$

Definition 1. The beta function, also called the Euler integral of the first kind, is a special function defined by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, x, y > 0,$$

and

$$\beta(a, x, y) = \int_0^a t^{x-1}(1-t)^{y-1}\mathrm{d}t, 0 < a < 1, x, y > 0,$$

is incomplete Beta function.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 2. Let $f \in L([a, b])$. The symbols $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ denote the left-side and right-side Riemann-Liouville integrals of the order α and are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t \quad (0 \leq a < x),$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad (0 < x < b),$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}\mathrm{d}t$ and $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

In the caes of $\alpha = 1$, the fractional integrals reduces to the classical integral. Recently, many authors have studied a number of inequalities by used the Riemann-Liouville fractional integrals, see [1-10,13,15-18] and the references cited therein.

Especially, in [3, 13], Imdat Işcan, Noor, and Awan proved a variant of Hermite-Hadamard-like type and Ostrowski-like type inequalities which hold for the convex functions via Riemann-Liouville fractional integrals.

Theorem 1.3. *Let $f : [a, b] \to R$ be twice differentiable function on (a, b) with $a < b$. If $f'' \in L([a, b])$ and $| f'' |$ is convex on $[a, b]$, then we have the following inequality for fractional integrals:*

$$\begin{aligned}&\Big|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big\{J^{\alpha}_{(\frac{a+b}{2})^-}f(a)+J^{\alpha}_{(\frac{a+b}{2})^+}f(b)\Big\}-f(\frac{a+b}{2})\Big|\\ &\le \frac{(b-a)^2}{2^4(\alpha+1)}\Big\{2^{\alpha+3}\beta(\frac{1}{2},\alpha+1,2)+\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+4)}\Big\}[| f''(a) |+| f''(b) |].\end{aligned}$$

Theorem 1.4. *Let $f : I \subseteq [0, \infty) \to R$ be a differentiable function on the interior I^0 of an interval I such that $f' \in L([a, b])$, where $a, b \in I^0$ with $a < b$. If $| f' |^q$ is s-convex on $[a, b]$ for some fixed $q \geq 1$, $x \in [a, b]$, $\mu \in [0, 1]$ and $\alpha > 0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\Big|(1-\mu)\Big\{\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\Big\}f(x)\\ &\quad+\mu\Big\{\frac{(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)}{b-a}\Big\}\\ &\quad-\frac{\Gamma(\alpha+1)}{b-a}\Big\{J^{\alpha}_{x^-}f(a)+J^{\alpha}_{x^+}f(b)\Big\}\Big|\\ &\le A_1^{1-\frac{1}{q}}(\alpha,\mu)\\ &\quad\times\Big[\frac{(x-a)^{\alpha+1}}{b-a}\Big\{ | f'(x) |^q A_2(\alpha,\mu,s)+ | f'(a) |^q A_3(\alpha,\mu,s)\Big\}^{\frac{1}{q}}\\ &\quad+\frac{(b-x)^{\alpha+1}}{b-a}\Big\{ | f'(x) |^q A_2(\alpha,\mu,s)+ | f'(b) |^q A_3(\alpha,\mu,s)\Big\}^{\frac{1}{q}}\Big],\end{aligned}$$

where

$$\begin{aligned}A_1(\alpha,\mu)&=\frac{2\alpha\mu^{1+\frac{1}{\alpha}}+1}{\alpha+1}-\mu,\\ A_2(\alpha,\mu,s)&=\frac{2\alpha\mu^{1+\frac{s+1}{\alpha}}+s+1}{(s+1)(\alpha+s+1)}-\frac{\mu}{s+1},\\ A_3(\alpha,\mu,s)&=\mu\Big(\frac{1-2(1-\mu^{\frac{1}{\alpha}})^{s+1}}{s+1}\Big)+\beta(\alpha+1,s+1)\\ &\qquad-2\beta(\mu^{\frac{1}{\alpha}},\alpha+1,s+1).\end{aligned}$$

In this paper, we give some generalized integral inequalities connected with the Simpson-like type for differentiable functions whose derivatives in the absolute value at certain powers are convex s-convex in the second sense via fractional integrals.

2 Lemmas

Now we turn our attention to establish integral inequalities of Simpson-like type inequality for convex functions via Riemann-Liouville fractional integrals, we need the lemmas below:

Lemma 1. *Let $f : I \subseteq R \to R$ be a differentiable function on the interior I^0 of an interval I such that $f' \in L([a,b])$, where $a, b \in I$ with $a < b$. Then, for any $0 \leq \lambda \leq 1$ and $n \geq 2$, the following identity holds:*

$$\begin{aligned} &I(f;\alpha;\lambda,n) \\ &\equiv \frac{1}{n}\Big\{\lambda f(a) + (1-\lambda)f(b)\Big\} + \big(1-\frac{1}{n}\big)f(\lambda a + (1-\lambda)b) \\ &\quad - \frac{\Gamma(\alpha+1)}{\lambda^{\alpha}(1-\lambda)^{\alpha}(b-a)^{\alpha}}\Big\{\lambda^{\alpha+1}J^{\alpha}_{x^-}f(a) + (1-\lambda)^{\alpha+1}J^{\alpha}_{x^+}f(b)\Big\} \\ &= \lambda(1-\lambda)(b-a)\Big[\int_0^1 (t^{\alpha} - \frac{1}{n})f'\big(tx + (1-t)a\big)\mathrm{d}t \\ &\quad + \int_0^1 (\frac{1}{n} - t^{\alpha})f'\big(tx + (1-t)b\big)\mathrm{d}t\Big], \end{aligned} \tag{3}$$

where $x = \lambda a + (1-\lambda)b$.

Proof. Integrating by parts and changing variable of definite integral, we have:

$$\begin{aligned} &\int_0^1 (t^{\alpha} - \frac{1}{n})f'\big(tx + (1-t)a\big)\mathrm{d}t \\ &= \frac{1}{(1-\lambda)(b-a)}\Big\{\frac{1}{n}f(a) + \big(1-\frac{1}{n}\big)f(x) - \frac{\Gamma(\alpha+1)}{(1-\lambda)^{\alpha}(b-a)^{\alpha}}J^{\alpha}_{x^-}f(a)\Big\}. \end{aligned} \tag{4}$$

Similarly, we have

$$\begin{aligned} &\int_0^1 (\frac{1}{n} - t^{\alpha})f'\big(tx + (1-t)b\big)\mathrm{d}t \\ &= \frac{1}{\lambda(b-a)}\Big\{\frac{1}{n}f(b) + \big(1-\frac{1}{n}\big)f(x) - \frac{\Gamma(\alpha+1)}{\lambda^{\alpha}(b-a)^{\alpha}}J^{\alpha}_{x^+}f(b)\Big\}. \end{aligned} \tag{5}$$

Multiplying both sides of (5) and (6) by $\frac{\lambda(1-\lambda)(b-a)}{\lambda}$ and $\frac{\lambda(1-\lambda)(b-a)}{1-\lambda}$, respectively, and adding the resulting two equalities we obtain the desired result.

Note that, if we choose $n = 3$ and $\lambda = \frac{1}{2}$, then we have

$$\begin{aligned}
&\frac{1}{6}\Big\{f(a) + 4f(\frac{a+b}{2}) + f(b)\Big\} \\
&\quad - \big(\frac{1}{2}\big)^{1-\alpha}\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big\{J_{(\frac{a+b}{2})^{-}}{}^{\alpha} f(a) + J^{\alpha}_{(\frac{a+b}{2})^{+}} f(b)\Big\} \\
&= \frac{b-a}{4}\Big[\int_0^1 (t^{\alpha} - \frac{1}{3}) f'\big(t\frac{a+b}{2} + (1-t)a\big)\mathrm{d}t \\
&\quad + \int_0^1 (\frac{1}{3} - t^{\alpha}) f'\big(t\frac{a+b}{2} + (1-t)b\big)\mathrm{d}t\Big]. \qquad (6)
\end{aligned}$$

Lemma 2. *For $0 \le \xi \le 1$, one has*

$$\begin{aligned}
&(a) \int_0^1 \big|\xi - t^{\alpha}\big|^q \mathrm{d}t \equiv \delta_1(\alpha, \xi, q) \\
&\quad = \frac{\xi^{q+\frac{1}{\alpha}}}{\alpha}\Big\{\beta(\frac{1}{\alpha}, 1+q) + \beta(-q - \frac{1}{\alpha}, 1+q) - \beta(\xi, -q - \frac{1}{\alpha}, 1+q)\Big\}, \\
&(b) \int_0^1 \big|\xi - t^{\alpha}\big|^q t\mathrm{d}t \equiv \delta_2(\alpha, \xi, q) \\
&\quad = \frac{\xi^{q+\frac{2}{\alpha}}}{\alpha}\Big\{\beta(\frac{2}{\alpha}, 1+q) + \beta(-q - \frac{2}{\alpha}, 1+q) - \beta(\xi, -q - \frac{2}{\alpha}, 1+q)\Big\}. \qquad (7)
\end{aligned}$$

Proof. These equalities follows from a straightfoward computation of definite integrals.

3 Some inequalities of Simpson-like type

Now we turn our attention to establish new integral inequalities of Hermite-Hadamard and Ostrowski type for convex functions via fractional integrals.

Theorem 3.1. *Let $f : I \subseteq R \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a, b])$, where $a, b \in I$ with $a < b$ and $\lambda \in [0, 1]$ and $n \ge 2$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned}
&\Big|I(f; \alpha; \lambda, n)\Big| \\
&\le \lambda(1-\lambda)(b-a) \\
&\quad \times \Big[\Big\{\delta_1(\alpha, \frac{1}{n}, 1) + (2\lambda - 1)\delta_2(\alpha, \frac{1}{n}, 1)\Big\}\big|f'(a)\big| \\
&\qquad + \Big\{\delta_1(\alpha, \frac{1}{n}, 1) + (1 - 2\lambda)\delta_2(\alpha, \frac{1}{n}, 1)\Big\}\big|f'(b)\big|\Big],
\end{aligned}$$

Proof. From Lemma 1, the convexity of $|\ f'\ |$ on $[a,b]$, and the noted power-mean integral inequality, we have

$$\begin{aligned}
&\Big|I(f;\alpha;\lambda,n)\Big| \\
&\le \lambda(1-\lambda)(b-a)\Big[\int_0^1 \big|t^\alpha-\frac{1}{n}\big|\big|f'\big(tx+(1-t)a\big)\big|\mathrm{d}t \\
&\quad+\int_0^1 \big|\frac{1}{n}-t^\alpha\big|\big|f'\big(tx+(1-t)b\big)\big|\mathrm{d}t\Big] \\
&\le \lambda(1-\lambda)(b-a)\Big[\Big(\int_0^1 \big|t^\alpha-\frac{1}{n}\big|(1-t)\mathrm{d}t\Big)\Big\{\big|f'(a)\big|+\big|f'(b)\big|\Big\} \\
&\quad+2\Big(\int_0^1 \big|t^\alpha-\frac{1}{n}\big|t\mathrm{d}t\Big)\big|f'\big(\lambda a+(1-\lambda)b\big)\big|\Big] \\
&=\lambda(1-\lambda)(b-a)\Big[2\delta_2(\alpha,\frac{1}{n},1)\big|f'\big(\lambda a+(1-\lambda)b\big)\big| \\
&\quad+\Big\{\delta_1(\alpha,\frac{1}{n},1)-\delta_2(\alpha,\frac{1}{n},1)\Big\}\Big\{\big|f'(a)\big|+\big|f'(b)\big|\Big\}\Big]. \qquad (8)
\end{aligned}$$

By Theorem 1.1 and the inequality (8), we have

$$\begin{aligned}
&\Big|I(f;\alpha;\lambda,n)\Big| \\
&\le \lambda(1-\lambda)(b-a)\Big[\Big\{\delta_1(\alpha,\frac{1}{n},1)+(2\lambda-1)\delta_2(\alpha,\frac{1}{n},1)\Big\}\big|f'(a)\big| \\
&\quad+\Big\{\delta_1(\alpha,\frac{1}{n},1)+(1-2\lambda)\delta_2(\alpha,\frac{1}{n},1)\Big\}\big|f'(b)\big|\Big].
\end{aligned}$$

Theorem 3.2. *Let $f: I\subseteq R\to R$ be a differentiable function on the interior I^0 of an interval I and $f'\in L([a,b])$, where $a,b\in I$ with $a<b$, $\lambda\in[0,1]$, and $n\ge 2$. If $|f'|^q$ is convex on $[a,b]$ for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\begin{aligned}
&\Big|I(f;\alpha;\lambda,n)\Big| \\
&\le \lambda(1-\lambda)(b-a)\delta_1^{\frac{1}{p}}(\alpha,\frac{1}{n},p) \\
&\quad\times\Big[\Big\{\frac{(1+\lambda)\big|f'(a)\big|^q+(1-\lambda)\big|f'(b)\big|^q}{2}\Big\}^{\frac{1}{q}} \\
&\quad+\Big\{\frac{\lambda\big|f'(a)\big|^q+(2-\lambda)\big|f'(b)\big|^q}{2}\Big\}^{\frac{1}{q}}\Big],
\end{aligned}$$

where $x=\lambda a+(1-\lambda)b$.

Proof. From Lemma 1, the convexity of $|\ f'\ |^q$ on $[a,b]$ for $q>1$ with

$\frac{1}{p}+\frac{1}{q}=1$, and Hölder integral inequality, we have

$$\begin{aligned}
&\Big|I(f;\alpha;\lambda,n)\Big| \\
&\le \lambda(1-\lambda)(b-a)\Big\{\int_0^1 \big|t^\alpha-\frac{1}{n}\big|^p \mathrm{d}t\Big\}^{\frac{1}{p}} \\
&\quad \times\Big[\Big\{\int_0^1 \big|f'\big(tx+(1-t)a\big)\big|^q \mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\qquad +\Big\{\int_0^1 \big|f'\big(tx+(1-t)b\big)\big|^q \mathrm{d}t\Big\}^{\frac{1}{q}}\Big]. \qquad (9)
\end{aligned}$$

By the convexity of $|\ f'\ |^q$ on $[a,b]$ for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\int_0^1 \big|f'\big(tx+(1-t)a\big)\big|^q \mathrm{d}t \le \frac{\big|f'(a)\big|^q+\big|f'(x)\big|^q}{2}, \qquad (10)$$

$$\int_0^1 \big|f'\big(tx+(1-t)b\big)\big|^q \mathrm{d}t \le \frac{\big|f'(x)\big|^q+\big|f'(b)\big|^q}{2}. \qquad (11)$$

By substituting (10) and (11) in (9), we get

$$\begin{aligned}
&\Big|I(f;\alpha;\lambda,n)\Big| \\
&\le \lambda(1-\lambda)(b-a)\delta_1^{\frac{1}{p}}(\alpha,\frac{1}{n},1) \\
&\quad \times\Big[\Big\{\frac{\big|f'(a)\big|^q+\big|f'(x)\big|^q}{2}\Big\}^{\frac{1}{q}}+\Big\{\frac{\big|f'(x)\big|^q+\big|f'(b)\big|^q}{2}\Big\}^{\frac{1}{q}}\Big]. \qquad (12)
\end{aligned}$$

By the inequality (8), we have

$$\big|f'(a)\big|^q+\big|f'(x)\big|^q \le (1+\lambda)\big|f'(a)\big|^q+(1-\lambda)\big|f'(b)\big|^q, \qquad (13)$$

$$\big|f'(x)\big|^q+\big|f'(b)\big|^q \le \lambda\big|f'(a)\big|^q+(2-\lambda)\big|f'(b)\big|^q. \qquad (14)$$

By substituting (13) and (14) in (12), we get the second inequality.

Theorem 3.3. *Let $f: I\subseteq R\to R$ be a differentiable function on the interior I^0 of an interval I and $f'\in L([a,b])$, where $a,b\in I$ with $a<b$, $\lambda\in[0,1]$, and $n\ge 2$. If $|f'|^q$ is convex on $[a,b]$ for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\begin{aligned}
&\Big|I(f;\alpha;\lambda,n)\Big| \\
&\le \lambda(1-\lambda)(b-a)\Big[\Big\{\Big(\delta_1(\alpha,\frac{1}{n},q)-(1-\lambda)\delta_2(\alpha,\frac{1}{n},q)\Big)\big|f'(a)\big|^q \\
&\qquad +(1-\lambda)\delta_2(\alpha,\frac{1}{n},q)\big|f'(b)\big|^q\Big\}^{\frac{1}{q}} \\
&\quad +\Big\{\Big(\lambda\delta_2(\alpha,\frac{1}{n},q)\big|f'(a)\big|^q+\Big(\delta_1(\alpha,\frac{1}{n},q)-\lambda\delta_2(\alpha,\frac{1}{n},q)\Big)\big|f'(b)\big|^q\Big\}^{\frac{1}{q}}\Big].
\end{aligned}$$

Proof. From Lemma 1, the convexity of $| f' |^q$ on $[a, b]$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and Hölder integral inequality, we have

$$
\begin{aligned}
&\Big|I(f;\alpha;\lambda,n)\Big| \\
&\le \lambda(1-\lambda)(b-a)\Big[\Big\{\int_0^1 \big|t^\alpha - \frac{1}{n}\big|^q \big|f'\big(tx+(1-t)a\big)\big|^q \mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\qquad + \Big\{\int_0^1 \big|t^\alpha - \frac{1}{n}\big|^q \big|f'\big(tx+(1-t)b\big)\big|^q \mathrm{d}t\Big\}^{\frac{1}{q}}\Big] \\
&\le \lambda(1-\lambda)(b-a)\Big[\Big\{\Big(\int_0^1 \big|t^\alpha - \frac{1}{n}\big|^q t\mathrm{d}t\Big)\big|f'\big(\lambda a+(1-\lambda)b\big)\big|^q \\
&\qquad + \Big(\int_0^1 \big|t^\alpha - \frac{1}{n}\big|^q (1-t)\mathrm{d}t\Big)\big|f'\big(a\big)\big|^q\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\Big(\int_0^1 \big|t^\alpha - \frac{1}{n}\big|^q t\mathrm{d}t\Big)\big|f'\big(\lambda a+(1-\lambda)b\big)\big|^q \\
&\qquad + \Big(\int_0^1 \big|t^\alpha - \frac{1}{n}\big|^q (1-t)\mathrm{d}t\Big)\big|f'\big(b\big)\big|^q\Big\}^{\frac{1}{q}}\Big] \\
&\le \lambda(1-\lambda)(b-a) \\
&\quad \times \Big[\Big\{\delta_2(\alpha,\frac{1}{n},q)\big|f'\big(x\big)\big|^q + \Big(\delta_1(\alpha,\frac{1}{n},q) - \delta_2(\alpha,\frac{1}{n},q)\Big)\big|f'\big(a\big)\big|^q\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\delta_2(\alpha,\frac{1}{n},q)\big|f'\big(x\big)\big|^q + \Big(\delta_1(\alpha,\frac{1}{n},q) - \delta_2(\alpha,\frac{1}{n},q)\Big)\big|f'\big(b\big)\big|^q\Big\}^{\frac{1}{q}}\Big]. \qquad (15)
\end{aligned}
$$

By the convexity of $| f' |^q$ on $[a, b]$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$
| f'(x) |^q = \big|f'\big(\lambda a + (1-\lambda)b\big)\big|^q \le \lambda\big|f'(a)\big|^q + (1-\lambda)\big|f'(b)\big|^q. \qquad (16)
$$

By substituting (16) in (15), we get the desired result.

Theorem 3.4. *Let $f : I \subseteq R \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a, b])$, where $a, b \in I$ with $a < b$, $\lambda \in [0, 1]$, and $n \ge 2$. If $|f'|^q$ is convex on $[a, b]$ for $q \ge 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$
\begin{aligned}
\Big|I(f;\alpha;\lambda,n)\Big| &\le \lambda(1-\lambda)(b-a)\delta_1^{\frac{1}{p}}(\alpha,\frac{1}{n},1) \\
&\times \Big[\Big\{\Big(\delta_1(\alpha,\frac{1}{n},1) - (1-\lambda)\delta_2(\alpha,\frac{1}{n},1)\Big)\big|f'\big(a\big)\big|^q
\end{aligned}
$$

$$+ (1-\lambda)\delta_2(\alpha, \frac{1}{n}, 1)\Big)\big|f'\big(b\big)\big|^q\Big\}^{\frac{1}{q}} + \Big\{\lambda\delta_2(\alpha, \frac{1}{n}, 1)\Big)\big|f'\big(a\big)\big|^q + \Big(\delta_1(\alpha, \frac{1}{n}, 1) - \lambda\delta_2(\alpha, \frac{1}{n}, 1)\Big)\big|f'\big(b\big)\big|^q\Big\}^{\frac{1}{q}}\Big].$$

Proof. Suppose that $q \geq 1$. By Lemma 1, the convexity of $|\ f'\ |^q$ on $[a, b]$, and the power-mean integral inequality, it follows that

$$\begin{aligned}
&\Big|I(f;\alpha;\lambda,n)\Big| \\
&\leq \lambda(1-\lambda)(b-a) \\
&\quad\times \Big[\Big\{\int_0^1 \big|t^\alpha - \frac{1}{n}\big|\mathrm{d}t\Big\}^{\frac{1}{p}}\Big\{\int_0^1 \big|t^\alpha - \frac{1}{n}\big|\big|f'\big(tx+(1-t)a\big)\big|^q\mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad+ \Big\{\int_0^1 \big|t^\alpha - \frac{1}{n}\big|\mathrm{d}t\Big\}^{\frac{1}{p}}\Big\{\int_0^1 \big|t^\alpha - \frac{1}{n}\big|\big|f'\big(tx+(1-t)b\big)\big|^q\mathrm{d}t\Big\}^{\frac{1}{q}}\Big] \\
&= \lambda(1-\lambda)(b-a)\delta_1^{\frac{1}{p}}(\alpha, \frac{1}{n}, 1) \\
&\quad\times \Big[\Big\{\int_0^1 \big|t^\alpha - \frac{1}{n}\big|\big|f'\big(t(\lambda a+(1-\lambda)b\big)+(1-t)a\big)\big|^q\mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad+ \Big\{\int_0^1 \big|t^\alpha - \frac{1}{n}\big|\big|f'\big(t(\lambda a+(1-\lambda)b\big)+(1-t)b\big)\big|^q\mathrm{d}t\Big\}^{\frac{1}{q}}\Big].
\end{aligned} \tag{17}$$

By the convexity of $|\ f'\ |^q$ on $[a, b]$, we have

$$\begin{aligned}
(i)\ &\int_0^1 \big|t^\alpha - \frac{1}{n}\big|\big|f'\big(t(\lambda a+(1-\lambda)b\big)+(1-t)a\big)\big|^q\mathrm{d}t \\
&\leq \Big(\delta_1(\alpha, \frac{1}{n}, 1) - (1-\lambda)\delta_2(\alpha, \frac{1}{n}, 1)\Big)\big|f'\big(a\big)\big|^q \\
&\quad+ (1-\lambda)\delta_2(\alpha, \frac{1}{n}, 1)\Big)\big|f'\big(b\big)\big|^q,
\end{aligned} \tag{18}$$

$$\begin{aligned}
(ii)\ &\int_0^1 \big|t^\alpha - \frac{1}{n}\big|\big|f'\big(t(\lambda a+(1-\lambda)b\big)+(1-t)b\big)\big|^q\mathrm{d}t \\
&\leq \lambda\delta_2(\alpha, \frac{1}{n}, 1)\big|f'\big(a\big)\big|^q + \Big(\delta_1(\alpha, \frac{1}{n}, 1) - \lambda\delta_2(\alpha, \frac{1}{n}, 1)\Big)\big|f'\big(b\big)\big|^q.
\end{aligned} \tag{19}$$

By substituting (18) and (19) in (17), we get the desired result.

Theorem 3.5. *Let $f : I \subseteq R \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a, b])$, where $a, b \in I$ with $a < b$ and $\lambda \in [0, 1]$ and $n \geq 2$. If $|f'|$ is s-convex in the second sense on $[a, b]$, then the following* *inequality holds:*

$$\begin{aligned}&\Big|I(f;\alpha;\lambda,n)\Big|\\&\le \lambda(1-\lambda)(b-a)\Big[\big\{2\lambda^s\delta_3(\alpha,n,s)+\delta_4(\alpha,n,s)\big\}\big|f'(a)\big|\\&\qquad+\big\{2(1-\lambda)^s\delta_3(\alpha,n,s)+\delta_4(\alpha,n,s)\big\}\big|f'(b)\big|\Big],\end{aligned}$$

where

$$\begin{aligned}\delta_3(\alpha,n,s)&=\frac{2\alpha n^{-\frac{s+\alpha+1}{\alpha}}+s+1}{(s+1)(s+\alpha+1)}-\frac{1}{n(s+1)},\\\delta_4(\alpha,n,s)&=\frac{1-2(1-(\frac{1}{n})^{\frac{1}{\alpha}}}{n(s+1)}+\beta(\alpha+1,s+1)\\&\quad-2\beta((\frac{1}{n})^{\frac{1}{\alpha}},\alpha+1,s+1).\end{aligned}$$

Proof. From Lemma 1, the s-convexity of $|\ f'\ |$ on $[a,b]$, we have

$$\begin{aligned}&\Big|I(f;\alpha;\lambda,n)\Big|\\&\le \lambda(1-\lambda)(b-a)\\&\quad\times\Big[\Big(\int_0^1\big|t^\alpha-\frac{1}{n}\big|t^s\mathrm{d}t\Big)\big|f'\big(\lambda a+(1-\lambda)b\big)\big|\\&\quad+\Big(\int_0^1\big|t^\alpha-\frac{1}{n}\big|(1-t)^s\mathrm{d}t\Big)\big|f'(a)\big|\\&\qquad\qquad+\Big(\int_0^1\big|t^\alpha-\frac{1}{n}\big|(1-t)^s\mathrm{d}t\Big)\big|f'(b)\big|\\&\quad+\Big(\int_0^1\big|t^\alpha-\frac{1}{n}\big|t^s\mathrm{d}t\Big)\big|f'\big(\lambda a+(1-\lambda)b\big)\big|\Big]\\&=\lambda(1-\lambda)(b-a)\\&\quad\times\Big[\Big(\int_0^1\big|t^\alpha-\frac{1}{n}\big|(1-t)^s\mathrm{d}t\Big)\Big\{\big|f'(a)\big|+\big|f'(b)\big|\Big\}\\&\quad+2\Big(\int_0^1\big|t^\alpha-\frac{1}{n}\big|t^s\mathrm{d}t\Big)\big|f'\big(\lambda a+(1-\lambda)b\big)\big|\Big]\\&\le \lambda(1-\lambda)(b-a)\Big[\big\{2\lambda^s\delta_3(\alpha,n,s)+\delta_4(\alpha,n,s)\big\}\big|f'(a)\big|\\&\qquad+\big\{2(1-\lambda)^s\delta_3(\alpha,n,s)+\delta_4(\alpha,n,s)\big\}\big|f'(b)\big|\Big].\end{aligned}$$

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