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Weak σ-Continuity on σ-Structures

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Abstract

We introduce to the notion of weakly σ-continuity on σ-structures and investigate characterizations for such continuity by using σ-semiopen, σ-preopen and σ-β-open. In particular, we introduce the notion of σ-regular open set and study characterizations of weakly σ-continuity by using σ-regular open sets.

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1 Introduction

The notions of generalized topology and generalized open sets are introduced by Császár [1] as the following: Let X be a nonempty set and μ be a collection of subsets of X. Then μ is called a *generalized topology* (briefly GT) on X iff

$\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $G = \cup_{i\in I} G_i \in \mu$. The elements of μ are called *g-open* sets and the complements are called *g-closed* sets. Kim and Min [2] introduced the notion of σ-structures which is an extended notion of generalized topology defined by Császár : $s \subseteq 2^X$ is called a *σ-structure* on X if for $i \in I \neq \emptyset$, $U_i \in s$ implies $\cup_{i\in I} U_i \in s$. The elements of s are called *σ-open* sets and the complements are called *σ-closed* sets. We also introduced the notions of the two operators i_s and c_s are defined as: $i_sA = \cup\{S \subseteq X : S \subseteq A,\ S$ is σ-open $\}$; $c_sA = \cap\{F \subseteq X : A \subseteq F,\ F$ is σ-closed $\}$.

Then we studied some properties for the notions and investigated the notion of σ-semiopen sets [3] in the spaces with σ-structures analogous to semi-open sets [6] introduced by Levine on a given topological space. We introduced the notions of σ-preopen sets [4] and σ-β-open sets [5]. In particular, we showed that the family $\sigma PO(X)$ (resp., $\sigma\beta O(X)$) of all σ-preopen sets (resp., σ-β-open sets) in (X, σ) is a generalized topology in sense of Császár. In [2], we also introduced the notion of σ-continuity [2] and obtained characterizations of σ-continuity by using the two operators i_s and c_s. The purpose of this paper is to study the extended notion of σ-continuity, and so we are going to introduce the notion of weak σ-continuity. Firstly, we introduce the weak σ-continuity and characterizations for the notion by using i_s and c_s. Then particularly, we investigate characterizations by using σ-semiopen sets, σ-preopen sets and σ-β-open sets. Secondly, we introduce the notion of σ-regular open set and study basic properties. Finally, we investigate characterizations of weakly σ-continuity by using σ-regular open sets.

2 Preliminaries

Theorem 2.1 ([2]). Let s be a σ-structure on a nonempty set X and $A, B \subseteq X$. Then

(1) $i_s\emptyset = \emptyset$ and $c_sX = X$.
(2) $i_sA \subseteq A$ and $A \subseteq c_sA$.
(3) If $A \subseteq B$, then $i_sA \subseteq i_sB$ and $c_sA \subseteq c_sB$.
(4) $i_si_sA = i_sA$ and $c_sc_sA = c_sA$.
(5) $c_s(A) = X - i_s(X - A)$ and $i_s(A) = X - c_s(X - A)$.
(6) A is σ-open iff $A = i_sA$ for $A \neq \emptyset$; A is σ-closed iff $A = c_sA$ for $A \neq X$.

Theorem 2.2 ([2]). Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then

(1) The collection $\mu = \{A \subseteq X : i_sA = A\}$ is a generalized topology on X.
(2) $x \in i_sA$ iff there exists a σ-open set S containing x such that $S \subseteq A$.
(3) $x \in c_sA$ iff $S \cap A \neq \emptyset$ for every σ-open set S containing x.

Let s, s' be σ-structures on X and Y, respectively. Then a function f :

$X \to Y$ is said to be *σ-continuous* if $f^{-1}(G) \in s$ for every $G \in s'$. Then f is σ-continuous iff $f^{-1}(i_s A) \subseteq i_s f^{-1}(A)$ for every $A \subseteq Y$.

3 Weak σ-continuous functions

Let σ be a σ-structure on a nonempty set X. The σ-structure σ is said to be *strong* [4] if $X \in \sigma$. From now on, unless otherwise specified, every σ-structure is strong. First, we introduce the notion of weak σ-continuous function on between two σ-structures s and s':

Definition 3.1. Let s, s' be σ-structures on X and Y, respectively. Then a function $f : X \to Y$ is said to be *weak σ-continuous* if for each $x \in X$ and each σ-open set V containing $f(x)$, there exists a σ-open set U containing x such that $f(U) \subseteq c_s(V)$.

Example 3.2. Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$. Consider two σ-structures $s = \{\{a\}, \{a, b\}, \{b, c\}, X\}$ and $s' = \{\{1, 3\}, \{2, 3\}, Y\}$ on Y, respectively. Define $f : (X, s) \to (Y, s')$ as follows: $f(a) = 2$, $f(b) = 2$, $f(c) = 3$. Then since $c_{s'}(\{1, 3\}) = c_{s'}(\{2, 3\}) = Y$, it is obvious that f is a weak σ-continuous function. But for a set $A = \{2, 3\}$, $f^{-1}(A) = \{a, c\}$ is not σ-open in s and so f is not σ-continuous.

Theorem 3.3. Let s, s' be σ-structures on X and Y, respectively. Let $f : X \to Y$ be a function. Then the following are equivalent:

(1) f is weakly σ-continuous.

(2) $f^{-1}(V) \subseteq i_s(f^{-1}(c_s(V)))$ for every σ-open subset V of Y.

(3) $c_s(f^{-1}(i_s(A))) \subseteq f^{-1}(A)$ for every σ-closed set A of Y.

(4) $c_s(f^{-1}(i_s(c_s(B)))) \subseteq f^{-1}(c_s(B))$ for every set B of Y.

(5) $f^{-1}(i_s(B)) \subseteq i_s(f^{-1}(c_s(i_s(B)))$ for every set B of Y.

(6) $c_s(f^{-1}(V)) \subseteq f^{-1}(c_s(V))$ for every σ-open subset V of Y.

Proof. (1) ⇒ (2) Let V be any σ-open subset of Y and $x \in f^{-1}(V)$. By weak σ-continuity, there exists a σ-open subset U of X containing x such that $f(U) \subseteq c_s(V)$. It implies that $x \in U \subseteq f^{-1}(c_s(V))$ and $x \in i(f^{-1}(c_s(V)))$. Hence $f^{-1}(V) \subseteq i(f^{-1}(c_s(V)))$.

(2) ⇒ (3) Let A be any σ-closed subset in Y. Then $Y - A$ is σ-open in Y and, by (2) and Theorem 2.1,

$$f^{-1}(Y - A) \subseteq i_s(f^{-1}(c_s(Y - A))) = i(f^{-1}(Y - i_s(A))) \subseteq X - c_s(f^{-1}(i_s(A))).$$

So $c_s(f^{-1}(i_s(A))) \subseteq f^{-1}(A)$.

(3) ⇒ (4) Let B be any subset of Y. Then from (3), it is obvious that $c_s(f^{-1}(i_s(c_s(B)))) \subseteq f^{-1}(c_s(B))$.

(4) $\Rightarrow$ (5) Let B be any subset of Y. From (4) and Theorem 2.1, it follows $f^{-1}(i_s(B)) = X - f^{-1}(c_s(Y - B)) \subseteq X - c_s(f^{-1}(i_s(c_s(Y - B)))) = i_s(f^{-1}(c_s(i_s(B))))$. Thus (5) is obtained.

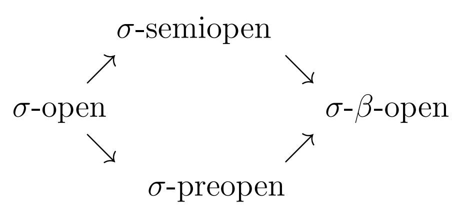
(5) $\Rightarrow$ (6) Let V be any σ-open subset of Y. Suppose $x \notin f^{-1}(c_s(V))$. Then $f(x) \notin c_s(V)$ and so there exists a σ-open set U containing $f(x)$ such that $U \cap V = \emptyset$ and so $c_s(U) \cap V = \emptyset$. By (5), $x \in f^{-1}(U) \subseteq i_s(f^{-1}(c_s(U)))$. From definition of i_s, there exists a σ-open set G containing x such that $x \in G \subseteq f^{-1}(c_s(U))$. Since $c_s(U) \cap V = \emptyset$ and $f(G) \subseteq c_s(U)$, we have $G \cap f^{-1}(V) = \emptyset$ and so $x \notin c_s(f^{-1}(V))$. Hence $c_s(f^{-1}(V)) \subseteq f^{-1}(c_s(V))$.

(6) $\Rightarrow$ (1) Let $x \in X$ and V any σ-open set in Y containing $f(x)$. From $V = i_s(V) \subseteq i_s(c_s(V))$, it follows $x \in f^{-1}(V) \subseteq f^{-1}(i_s(c_s(V))) = X - f^{-1}(c_s(Y - c_s(V))) \subseteq X - c_s(f^{-1}(Y - c_s(V))) = i_s(f^{-1}(c_s(V)))$. Then there exists a σ-open subset U containing x in X such that $U \subseteq f^{-1}(c_s(V))$. Hence f is weakly σ-continuous. $\square$

Now we are going to investigate characterizations by using σ-semiopen sets, σ-preopen sets and σ-β-open sets. First, we recall the notions of such generalized σ-open sets:

Definition 3.4 ([3,4,5]). Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then A is said to be

(1) σ-*semiopen* [3] if $A \subseteq c_s(i_s(A))$,
(2) σ-*preopen* [4] if $A \subseteq i_s(c_s(A))$,
(3) σ-β-*open* [5] if $A \subseteq c_s(i_s(c_s(A)))$.



Theorem 3.5. Let s, s' be σ-structures on X and Y, respectively. Then the following are equivalent:

(1) f is weakly σ-continuous.
(2) $c_s(f^{-1}(i_s(c_s(G)))) \subseteq f^{-1}(c_s(G))$ for every σ-β-open set G of Y.
(3) $c_s(f^{-1}(i_s(c_s(G)))) \subseteq f^{-1}(c_s(G))$ for every σ-semiopen set G of Y.

Proof. (1) $\Rightarrow$ (2) Let G be any σ-β-open set. Then $i_s(c_s(G)) \subseteq i_s(c_s(i_s(c_s(G))))$. For $c_s(i_s(c_s(G)))$, from (4) of Theorem 3.3, it follows $c_s(f^{-1}(i_s(c_s(G)))) \subseteq c_s(f^{-1}(i_s(c_s(i_s(c_s(G)))))) \subseteq f^{-1}(c_s(i_s(c_s(G)))) \subseteq f^{-1}(c_s(G))$.

(2) $\Rightarrow$ (3) Since every σ-semiopen set is σ-β-open, it is obvious.

(3) $\Rightarrow$ (1) Let V be any σ-open subset of Y. Then from hypothesis, we have $c_s(f^{-1}(V)) \subseteq c_s(f^{-1}(i_s(c_s(V)))) \subseteq f^{-1}(c_s(V))$. So by (6) of Theorem 3.5, f is weakly σ-continuous. $\square$

Theorem 3.6. Let s, s' be σ-structures on X and Y, respectively. Then the following are equivalent:

(1) f is weakly σ-continuous.

(2) $c_s(f^{-1}(i_s(c_s(G)))) \subseteq f^{-1}(c_s(G))$ for every σ-preopen set G of Y.

(3) $c_s(f^{-1}(G)) \subseteq f^{-1}(c_s(G))$ for every σ-preopen set G of Y.

(4) $f^{-1}(G) \subseteq i_s(f^{-1}(c_s(G)))$ for every σ-preopen set G of Y.

Proof. (1) $\Rightarrow$ (2) Let G be any σ-preopen set in Y. Then $c_s(G)$ is σ-closed. So from (3) of Theorem 3.3, it is obvious that $c_s(f^{-1}(i_s(c_s(G)))) \subseteq f^{-1}(c_s(G))$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) Let G be any σ-preopen set in Y. Then from definition of σ-preopen sets and (3), it follows that $f^{-1}(G) \subseteq f^{-1}(i_s(c_s(G))) = f^{-1}(Y - c_s(Y - c_s(G))) \subseteq X - f^{-1}(c_s(Y - c_s(G))) \subseteq X - c_s(f^{-1}(Y - c_s(G))) = X - c_s(X - f^{-1}(c_s(G))) = i_s(f^{-1}(c_s(G)))$. Hence (4) is obtained.

(4) $\Rightarrow$ (1) Since every σ-open set is σ-preopen, from hypothesis and (2) of Theorem 3.3, f is weakly σ-continuous. $\square$

Definition 3.7. Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then A is said to be *σ-regular open* if $A = i_s(c_s(A))$. The complement of σ-regular open set is called a *σ-regular closed set*.

Remark 3.8. Every σ-regular open set is clearly σ-open but the converse is not true in general as shown in the next example.

Example 3.9. Let $X = \{a, b, c\}$ and a σ-structure $s = \{\{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then for a σ-open set $\{b\}$, $i_s(c_s(\{b\})) = i_s(\{b, c\}) = \{b, c\}$, and so the σ-open set $\{b\}$ is not σ-regular open.

The following diagram is obtained:

$$\begin{array}{ccccccc} & & & & \sigma\text{-semiopen} & & \\ & & & \nearrow & & \searrow & \\ \sigma\text{-regular open} & \rightarrow & \sigma\text{-open} & & & & \sigma\text{-}\beta\text{-open} \\ & & & \searrow & & \nearrow & \\ & & & & \sigma\text{-preopen} & & \end{array}$$

Lemma 3.10. Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then σ-regular closed if and only if $A = c_s(i_s(A))$.

Proof. Straightforward. $\square$

Remark 3.11. Let s be a σ-structure on a nonempty set X. The intersection (union) of two σ-regular open sets is not always σ-regular open.

Example 3.12. Let $X = \{a, b, c\}$ and $s = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ a σ-structure:

(1) Then both $\{a, c\}$ and $\{b, c\}$ are σ-regular open. But $i_s(c_s(\{a, c\} \cap \{b, c\})) = i_s(c_s(\{c\})) = i_s(\{c\}) = \emptyset \neq \{a, c\} \cap \{b, c\}$, so the intersection is not σ-regular open.

(2) Consider two σ-regular open sets $\{a\} = i_s(c_s(\{a\}))$ and $\{b\} = i_s(c_s(\{b\}))$. For $\{a, b\}$, $c_s(\{a, b\}) = X$ and $i_s(c_s(\{a, b\})) = X$, and so the union of two σ-regular open sets $\{a\}$ and $\{b\}$ is not σ-regular open.

Theorem 3.13. Let s be a σ-structure on a nonempty set X and $A \subseteq X$. If A is

σ-closed, then $i_s(A)$ is σ-regular open.

Proof. Let A be a σ-closed set. Then $i_s(A) = i_s(i_s(A)) \subseteq i_s(c_s(i_s(A))) \subseteq i_s(c_s(A)) = i_s(A)$. So $i_s(A) = i_s(c_s(i_s(A)))$, and $i_s(A)$ is σ-regular open. □

Corollary 3.14. Let s be a σ-structure on a nonempty set X and $A \subseteq X$. If A is

σ-open, then $c_s(A)$ is σ-regular closed.

Theorem 3.15. Let s, s' be σ-structures on X and Y, respectively. Let $f : X \to Y$ be a function. Then the following are equivalent:

(1) f is weakly σ-continuous.

(2) $c_s(f^{-1}(i_s(F))) \subseteq f^{-1}(F)$ for every σ-regular closed subset F of Y.

Proof. (1) $\Rightarrow$ (2) Let F be any σ-regular closed subset of Y. Then since F is σ-closed, by (3) of Theorem 3.3, it is obviously $c_s(f^{-1}(i_s(F))) \subseteq f^{-1}(F)$.

(2) $\Rightarrow$ (1) Let V be any σ-open subset of Y. Then the above corollary, $c_s(V)$ is σ-regular closed, and by hypothesis, $c_s(f^{-1}(V)) \subseteq c_s(f^{-1}(i_s(c_s(V))) \subseteq f^{-1}(c_s(V))$. Hence by (6) of Theorem 3.3, f is weakly σ-continuous. □

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