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Equivalence between Semimartingales and Itô Processes

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Abstract

This paper is devoted to the determination of these semimartingales which are uniquely represented as Itô processes. This allows for the extension of the Girsanov Theorem.

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In the rest of this article, we assume a complete, filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{F} = \mathcal{F}_T$. The filtration is supposed to satisfy the following 'usual consitions' ([3, Def.1.1]) :

(a) $\mathcal{F}_t = \mathcal{F}_{t^+} = \cap_{u>t} F_u, t \in [0,T]$ (right -continuity)

(b) $\mathcal{F}_0$ contains the $\mathbb{P}$ -null probability sets (completenss).

1 On Equivalence between Semimartingale and Itô processes

The main results of this paper refer to whether a semimartingale in the sense of is uniquely represented as an Itô procsses. Of course, the opposite assertion is obvious -that an Itô process is a semimartingale. Actually, we start with proving that if A^+, A^- are RCLL processes, then every path function ($\mathbb{P}$-a.e.) is a differentiable function, since it is increasing and it has at most countable number of discontinuites. After that, we prove the general result. The Girsanov Theorem is also extended under this frame.

First of all we recall some notions, whose detailed definitions are useful for what follows.

Definition 1. *A stochastic process $X = (X_t)_{t\in[0,T]}$ such that $X_t : \Omega \to \mathbb{R}$ is called $\mathbb{F}$-**progressively measurable** if and only if for any $t \in [0,T]$ and $A \in \mathcal{B}(\mathbb{R})$, where the last notation denotes the Borel σ-algebra of $\mathbb{R}$, the set $\{(s,\omega) : 0 \leq s \leq t, X_s(\omega) \in A\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t$.*

Definition 2. *(see also [2, Def.3.1]) A **Semimartingale** X is any $\mathbb{F}$ -adapted process of the form $X = X_0 + A^+ - A^- + M$, $\mathbb{P}$-a.e. Specifically, A^+, A^- denote the **positive variation** and the **negative variation processs of** A, respectively. M is a **local martingale** M. For the corresponding random variables, we have $X_t = X_0 + A_t^+ - A_t^- + M_t$, for any $t \in [0,T]$, $\mathbb{P}$-a.e.*

Definition 3. *(see also [2, Def.5.15] A **local martingale** $M = (M_t)_{t\in[0,T]}$ is an $\mathbb{F}$-adapted stochastic process, such that a sequence of stopping -times $(\tau_k)_{k\in\mathbb{N}}$ exists such that*

1. *$\mathbb{P}(lim_{k\to+\infty}\tau_k = T) = 1$,*
2. *the stopped process $(M_{t\wedge\tau_k}, \mathbb{F})$ is a martingale for any $k \geq 1$.*

If moreover the process M is continuous, we say that this is a continuous local martingale.

Definition 4. *A continuous martingale M is an $\mathbb{F}$-martingale for which any sample path is continuous $\mathbb{P}$-a.e.*

Definition 5. *Suppose that β is an $\mathbb{F}$-adapted process. The **positive variation process** β^+ of β is defined as follows:*

$$\beta_t^+ = \sup\{\sum_{i=1}^{n}(\beta_{t_i} - \beta_{t_{i-1}})^+ : \{t_0, t_1, ..., t_n\} \in P(0,t)\},$$

P-a.e., where $P(0,t)$ denotes the set of partitions of the interval $[0,t]$, for any $t \in (0,T]$.

Definition 6. *The **negative variation process** β^- of β is defined as follows:*

$$\beta_t^- = \sup\{\sum_{i=1}^{n}(\beta_{t_i} - \beta_{t_{i-1}})^- : \{t_0, t_1, ..., t_n\} \in P(0,t)\},$$

P-a.e., where $P(0,t)$ denotes the set of partitions of the interval $[0,t]$, for any $t \in (0,T]$.

Definition 7. *The process A in the above representation is a* ***locally bounded variation process****, if and only if for A^+, A^-,*

$$A^+_s, A^-_s \in BV_0[0,t], t \in (0,T].$$

These definitions are given according to what is mentioned in [1, Ch.8.6]. According to what is mentioned in this Chapter, for any $t \in (0,T]$, we may suppose that $\mathbb{P}$-a.e. the random variables A^+_t, A^-_t belong to the AL -space $BV_0[0,t]$, because they have bounded positive and negative variation with respect to $s \in [0,t]$. The total variation of the process A in this case is $|A|_s = A^+_s + A^-_s, s \in [0,t]$. Also, the total variation norm in this space which is an L -norm and makes it a Banach lattice ([1, Th.8.44]) is equal to $|A|_t(\Omega), t \in (0,T]$. We have to mention here that $BV_0[0,t]$ is a function space concerning the paths of A^+, A^- and not a stochastic process space.

Lemma 8. *If A is a process of locally bounded variation, the paths of $A^+_s, A^-_s, s \in [0,t], t \in (0,T]$ may be taken to be differentiable functions with respect to s, except a set of $\lambda_{[0,T]} \otimes \mathbb{P}$ -measure zero, where $\lambda_{[0,T]}$ denotes the Lebesgue measure of $[0,T]$, also considered as a topological space.*

Proof: The paths of A^+, A^- are increasing functions with respect to $s \in [0,t], t \in (0,T]$, hence they have at least countable points of discontinuity with respect to t, hence an *indistigushable* (see [3, p.4]) pair of stochastic processes of both A^+, A^- with Right- Continuous-having-Left Limit paths exist. However, according to [3, Th.6.1, p.4], every path of both A^+, A^- at each interval $[0,t], t \in (0,T]$ has only finite number of discontinuites. This implies that the derivative of A^+_t, A^-_t with respect to t is well-defined, $\lambda \otimes \mathbb{P}$ -a.e. If we denote these path-derivatives by a^+_t, a^-_t we get that

$$A^+_t = \int_0^t a^+_s ds, A^-_t = \int_0^t a^-_s ds, \ \lambda_{[0,T]} \otimes \mathbb{P} - a.e.$$

Proposition 9. *(see also [2, Prob. 4.16]) Let $W = (W_t)_{t \in [0,T]}$ be an one -dimensional $\mathbb{F}$-Brownian motion, while $M = (M_t)_{t \in [0,T]}$ is a local martingale with respect to $\mathbb{F}$, such that $M_0 = 0$ and its paths are continuous, $\mathbb{P}$-a.e. Then a progressively measurable process $Y = (Y_t)_{t \in [0,T]}$ exists, such that $\mathbb{E}(\int_0^T Y_t^2 dt) < \infty$ and*

$$M_t = \int_0^t Y_s dW_s,$$

$\mathbb{P}$-a.e. Moreover, Y is adapted to the filtration $\mathbb{F}$, which is the augmentation under $\mathbb{P}$ of the filtration $\mathbb{F}^W$, generated by W.

Proof: Since $(M_t)_{t \in [0,T]}$ is a local $\mathbb{F}$- martingale, a sequence $(\tau_k)_{k \in \mathbb{N}}$ of $\mathcal{F}$-stopping times exists, such that $\mathbb{P}(\lim_{k \to \infty} \tau_k = T) = 1$ and $M_{t \wedge \tau_k}$ is an $\mathbb{F}$

-martingale for any $k \in \mathbb{N}$. We notice that $M_{t\wedge\tau_k} \to M_t$, $\mathbb{P}$-a.e. for any $t \in [0,T]$, which implies $M_{t\wedge\tau_k} \xrightarrow{L^1} M_t$, hence we also get that $M_{t\wedge\tau_k} \xrightarrow{L^2} M_t$. The first implication is due to the Dominated Convergence Theorem, while the second one is due to the Hölder Inequality. In the first implication we use the fact that M has continuous paths, hence $|M_t| \leq L_T = M_T^+ + M_T^-$, which holds for any $t \in [0,T]$, $\mathbb{P}$-a.e. We notice that according to the definition of positive and negative variation of M, L_T is a constant random variable, hence integrable in the space $L^1(\Omega, \mathcal{F}, \mathbb{P})$. It is obvious that since M has continuous paths $\mathbb{P}$-a.e (by definition), the positive and the negative variation processes M^+, M^- respectively, are well-defined. But the latest convergence result implies that any such M is then a squarely integrable martingale, which is also continuous by definition. Since this is true, from the Representation of Square-Integrable Martingales [2, Th.4.15], we obtain the final implication.

The next theorem is the essential results of this part, arising from Proposition 9 and Lemma 8.

Theorem 10. *Given an (one-dimensional)* $\mathbb{F}$*-Brownian motion, any* $\mathbb{F}$*- semimartingale* X *such that* A^+, A^- *are either of* ***locally bounded variation*** *or in a more general manner* λ***-integrable****-namely*

$$A_t^+ = \int_0^t a^+(s)ds, A_t^- = \int_0^t a^-(s)ds, t \in [0,T]$$

are well-defined, $\mathbb{P}$*-a.e., while in the representation* $X = X_0 + A_t + M_t, t \in [0,T]$, M *is a continuous local martingale, then* X *is an Itô process, with respect to the relevant Brownian motion filtration.*

If we recall the extended Itô formula, also mentioned as Kunita -Watanabe formula for continuous semimartingales. Suppose that $X = (X_t)_{t\in T}$ is a continuous $\mathbb{F}$-adapted semimartingale, which has a decomposition $X_t = X_0 + M_t + A_t$, for any $t \in [0,T]$, $\mathbb{P}$-a.e. $M = (M_t)_{t\in[0,T]}$ is a continuous local martingale, and $A = (A_t)_{t\in[0,T]}$ is an $\mathbb{F}$- adapted process, which is the difference of the positive and negative variation processes A^+, A^-, respectively. Then, if $f : \mathbb{R} \to \mathbb{R}$ a C^2-function, this formula is the following:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dA_s + \frac{1}{2}\int_0^t f''(X_s)d < M >_s,$$

see also [2, Th.3.3]. If we put $f(x) = x$, we simply take the above decomposition of X again.

The following Theorem is also direct.

Theorem 11. *Given an (one-dimensional)* $\mathbb{F}$*-Brownian motion, any* $\mathbb{F}$*- semimartingale* X *such that* A^+, A^- *are either of* ***locally bounded variation*** *or* *in a more general manner* λ***-integrable****-namely*

$$A_t^+ = \int_0^t a^+(s)ds, A_t^- = \int_0^t a^-(s)ds, t \in [0,T]$$

are well-defined, $\mathbb{P}$*-a.e., while in the representation* $X = X_0 + A_t + M_t, t \in [0,T]$, M *is a continuous local martingale, the Girsanov-Cameron-Martin Theorem is applicable, if the Novikov condition* $\mathbb{E}(e^{\int_0^T X_t^2 dt}) < \infty$ *holds.*

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