International Journal of Mathematical Analysis
Vol. 9, 2015, no. 15, 729 - 734
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijma.2015.5227

Some Identities of Symmetry for Generalized Carlitz-type q-Euler Polynomials under the Symmetric Group of Degree Four

Dae San Kim

Department of Mathematics, Sogang University
Seoul 121-742, Republic of Korea

Dmitry V. Dolgy

School of Natural Sciences
Far Eastern Federal University, Vladivostok, Russia

Taekyun Kim

Department of Mathematics, Kwangwoon University
Seoul 139-701, Republic of Korea

Abstract

In this paper, we give some identities of symmetry for the generalized Carlitz-type q-Euler polynomials under symmetric group of degree four which are derived from the fermionic p-adic q-integrals on $\mathbb{Z}_p$.

Mathematics Subject Classification: 11B68, 11S80, 05A19, 05A30

Keywords: identities of symmetry, generalized Carlitz-type q-Euler polynomial, fermionic p-adic integral, symmetric group of degree four

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $|\cdot|_p$ be the normalized p-adic absolute value with $|p|_p = \frac{1}{p}$, and let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-number of x is deifned as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the ferminonic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$\begin{aligned} I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \\ &= \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \quad (\text{see } [3,\ 4,\ 6,\ 7]). \end{aligned} \tag{1.1}$$

For $d \in \mathbb{N}$ with $(d,p)=1$ and $d \equiv 1 \pmod 2$, we set

$$X = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \quad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

and

$$a + dp^N\mathbb{Z}_p = \left\{ x \in X \,|\, x \equiv a \pmod{dp^N} \right\},$$

where $a \in \mathbb{Z}$ lies $0 \le a < dp^N$.

For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, let χ be a Dirichlet character with conductor d. It is well known that the generalized Euler polynomial attached to χ are defined by the generating function to be

$$\sum_{n=0}^{\infty} E_{n,\chi}(x) \frac{t^n}{n!} = \frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a e^{at}}{e^{dt}+1} e^{xt}, \quad (\text{see } [1\text{–}7]). \tag{1.2}$$

When $x = 0$, $E_{n,\chi} = E_{n,\chi}(0)$ are called the Euler numbers attached to χ. If $\chi = \chi^0$ is the trivial character, then $E_n(x) = E_{n,\chi^0}(x)$ are called the Euler polynomials.

In [8], the generalized Carlitz-type q-Euler polynomials attached to χ are defined as

$$E_{n,\chi,q}(x) = \int_X \chi(y)\, [x+y]_q^n \, d\mu_{-q}(y), \quad (n \ge 0). \tag{1.3}$$

Note that

$$\lim_{q\to 1} E_{n,\chi,q}(x) = \lim_{q\to 1} \int_X \chi(y)\, [x+y]_q^n \, d\mu_{-q}(y)$$

$$= \int_X \chi(y)(x+y)^n \, d\mu_{-1}(y)$$
$$= E_{n,\chi}(x).$$

When $x=0$, $E_{n,\chi,q} = E_{n,\chi,q}(0)$ are called the generalized Carlitz-type q-Euler numbers attached to χ.

The purpose of this paper is to investigate symmetric properties and identities for the generalized Carlitz-type q-Euler polynomials attached to χ under the symmetric group of degree four arising from fermionic p-adic q-integrals on $\mathbb{Z}_p$.

2. Symmetry identities of the generalized Carlitz-type q-Euler polynomials attached to χ under S_4

Let $w_1, w_2, w_3, w_4 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, $w_3 \equiv 1 \pmod 2$, $w_4 \equiv 1 \pmod 2$. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, let χ be a Dirichlet character with conductor d. From (1.1) and (1.3),

$$\int_X \chi(y) e^{[w_1w_2w_3y + w_1w_2w_3w_4x + w_4w_ww_3i + w_4w_1w_3j + w_4w_1w_2k]_q t} d\mu_{-q^{w_1w_2w_3}}(y) \tag{2.1}$$
$$= \lim_{N\to\infty} \frac{1}{[dw_4p^N]_{-q^{w_1w_2w_3}}} \sum_{l=0}^{dw_4-1} \sum_{y=0}^{p^N-1} \chi(l)(-1)^{l+y}$$
$$\times e^{[w_1w_2w_3(l+dw_4y) + w_1w_2w_3w_4x + w_4w_2w_3i + w_4w_1w_3j + w_4w_1w_2k]_q t} q^{w_1w_2w_3(l+dw_4y)}.$$

By (2.1), we get

$$\frac{1}{[2]_{q^{w_1w_2w_3}}} \sum_{i=0}^{dw_1-1} \sum_{j=0}^{dw_2-1} \sum_{k=0}^{dw_3-1} (-1)^{i+j+k} q^{w_4w_2w_3i + w_4w_1w_3j + w_4w_1w_2k} \chi(i)\chi(j)\chi(k) \tag{2.2}$$
$$\times \int_X \chi(y) e^{[w_1w_2w_3y + w_1w_2w_3w_4x + w_4w_2w_3i + w_4w_1w_3j + w_4w_1w_2k]_q t} d\mu_{-q^{w_1w_2w_3}}(y)$$
$$= \lim_{N\to\infty} \frac{1}{1+q^{dw_1w_2w_3w_4p^N}}$$
$$\times \sum_{i=0}^{dw_1-1} \sum_{j=0}^{dw_2-1} \sum_{k=0}^{dw_3-1} \sum_{l=0}^{dw_4-1} (-1)^{i+j+k+l} q^{w_4w_2w_3i + w_4w_1w_3j + w_4w_1w_2k + w_1w_2w_3l} \chi(ijkl)$$
$$\times \sum_{y=0}^{p^N-1} (-q)^{dw_1w_2w_3w_4y} \chi(y) e^{[w_1w_2w_3(l+w_4y) + w_1w_2w_3w_4x + w_4w_2w_3i + w_4w_1w_3j + w_4w_1w_2k]_q t}.$$

Note that the equation (2.2) is invariant for any permutation $\sigma \in S_4$. Thus, we have the following theorem.

Theorem 2.1. *Let $w_1, w_2, w_3, w_4 \in \mathbb{N}$, with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, $w_3 \equiv 1 \pmod 2$, $w_4 \equiv 1 \pmod 2$. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, let χ be a Dirichlet character with conductor d. Then, the following expressions*

$$\frac{1}{[2]_{q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}} \sum_{i=0}^{dw_{\sigma(1)}-1} \sum_{j=0}^{dw_{\sigma(2)}-1} \sum_{k=0}^{dw_{\sigma(3)}-1} (-1)^{i+j+k}$$
$$\times q^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k} \chi(i)\chi(j)\chi(k)$$
$$\times \int_X e^{[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}y+w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}x+w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k]_q t}$$
$$\times \chi(y)\, d\mu_{-q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}(y)$$

are the same for any $\sigma \in S_4$.

It is not difficult to show that

$$\int_X \chi(y)\, e^{[w_1w_2w_3y+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t} d\mu_{-q^{w_1w_2w_3}}(y) \tag{2.3}$$
$$= \sum_{n=0}^{\infty} [w_1w_2w_3]_q^n \int_X \chi(y) \left[y + w_4x + \frac{w_4}{w_1}i + \frac{w_4}{w_2}j + \frac{w_4}{w_3}k\right]_{q^{w_1w_2w_3}}^n d\mu_{-q^{w_1w_2w_3}}(y)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} [w_1w_2w_3]_q^n E_{n,\chi,q^{w_1w_2w_3}}\left(w_4x + \frac{w_4}{w_1}i + \frac{w_4}{w_2}j + \frac{w_4}{w_3}k\right)\frac{t^n}{n!}.$$

Thus, by (2.3), we get

$$\int_\chi \chi(y)\left[w_1w_2w_3y + w_4w_2w_3w_1x + w_4w_2w_3i + w_4w_1w_3j + w_4w_1w_2k\right]_q^n d\mu_{-q^{w_1w_2w_3}}(y) \tag{2.4}$$
$$= [w_1w_2w_3]_q^n E_{n,\chi,q^{w_1w_2w_3}}\left(w_4x + \frac{w_4}{w_1}i + \frac{w_4}{w_2}j + \frac{w_4}{w_3}k\right), \quad (n \geq 0).$$

Therefore, by Theorem 2.1 and (2.4), we obtain the following theorem.

Theorem 2.2. *Let $n \geq 0$, $w_1, w_2, w_3, w_4, d \in \mathbb{N}$, with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, $w_3 \equiv 1 \pmod 2$, $w_4 \equiv 1 \pmod 2$, $d \equiv 1 \pmod 2$, and let χ be a Dirichlet character with conductor d. Then, the following expressions*

$$\frac{\left[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}\right]_q^n}{[2]_{q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}}$$
$$\times \sum_{i=0}^{dw_{\sigma(1)}-1} \sum_{j=0}^{dw_{\sigma(2)}-1} \sum_{k=0}^{dw_{\sigma(3)}-1} (-1)^{i+j+k} q^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k}$$
$$\times \chi(i)\chi(j)\chi(k) E_{n,\chi,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}\left(w_{\sigma(4)}x + \frac{w_{\sigma(4)}}{w_{\sigma(1)}}i + \frac{w_{\sigma(4)}}{w_{\sigma(2)}}j + \frac{w_{\sigma(4)}}{w_{\sigma(3)}}k\right)$$

are the same for any $\sigma \in S_4$.

We observe that

$$\left[y+w_4x+\frac{w_4}{w_1}i+\frac{w_4}{w_2}j+\frac{w_4}{w_3}k\right]^n_{q^{w_1w_2w_3}} \tag{2.5}$$

$$=\sum_{m=0}^{n}\binom{n}{m}\left(\frac{[w_4]_q}{[w_1w_2w_3]_q}\right)^{n-m}[w_2w_3i+w_1w_3j+w_1w_2k]^{n-m}_{q^{w_4}} \tag{2.6}$$
$$\times q^{m(w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k)}$$
$$\times [y+w_4x]^m_{q^{w_1w_2w_3}}\,.$$

Thus, by (1.3) and (2.5), we get

$$\frac{[w_1w_2w_3]^n_q}{[2]_{q^{w_1w_2w_3}}}\sum_{i=0}^{dw_1-1}\sum_{j=0}^{dw_2-1}\sum_{k=0}^{dw_3-1}(-1)^{i+j+k}\,q^{w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k}\chi(i)\,\chi(j)\,\chi(k) \tag{2.7}$$
$$\times\int_{\mathbb{Z}_p}\left[y+w_4x+\frac{w_4}{w_1}i+\frac{w_4}{w_2}j+\frac{w_4}{w_3}k\right]^n_{q^{w_1w_2w_3}}\chi(y)\,d\mu_{-q^{w_1w_2w_3}}(y)$$
$$=\sum_{m=0}^{n}\binom{n}{m}\frac{[w_1w_2w_3]^m_q}{[2]_{q^{w_1w_2w_3}}}[w_4]^{n-m}_q\,E_{m,\chi,q^{w_1w_2w_3}}(w_4x)\,\hat{T}_{w,\chi,q^{w_4}}(w_1,w_2,w_3\mid m)\,,$$

where

$$\hat{T}_{n,\chi,q}(w_1,w_2,w_3\mid m) \tag{2.8}$$
$$=\sum_{i=0}^{dw_1-1}\sum_{j=0}^{dw_2-1}\sum_{k=0}^{dw_3-1}(-1)^{i+j+k}\,\chi(ijk)\,q^{(m+1)(w_2w_3i+w_1w_3j+w_1w_2k)}$$
$$\times[w_2w_3i+w_1w_3j+w_1w_2k]^{n-m}_q\,.$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.3. *Let $n\geq 0$, $w_1,w_2,w_3,w_4,d\in\mathbb{N}$ with $w_1\equiv 1\pmod 2$, $w_2\equiv 1\pmod 2$, $w_3\equiv 1\pmod 2$, $w_4\equiv 1\pmod 2$, $d\equiv 1\pmod 2$, and let χ be a Dirichlet character with conductor d. Then, the following expressions*

$$\sum_{m=0}^{n}\binom{n}{m}\frac{\left[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}\right]^m_q}{[2]_{q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}}\left[w_{\sigma(4)}\right]^{n-m}_q$$
$$\times E_{m,\chi,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}\left(w_{\sigma(4)}x\right)\hat{T}_{n,\chi,q^{w_{\sigma(4)}}}\left(w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}\mid m\right)$$

are the same for any $\sigma\in S_4$.

References

[1] L. Carlitz, *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc. **76** (1954), 332–350. MR 0060538 (15,686a). http://dx.doi.org/10.1090/s0002-9947-1954-0060538-2

[2] D. V. Dolgy, D. S. Kim, T. G. Kim, and J. J. Seo, *Identities of symmetry for higher-order generalized q-Euler polynomials*, Abstr. Appl. Anal. (2014), Art. ID 286239, 6. MR 3176733. http://dx.doi.org/10.1155/2014/286239

[3] D. S. Kim and T. Kim, *Identities of symmetry for generalized q-Euler polynomials arising from multivariate fermionic p-adic integral on* $\mathbb{Z}_p$, Proc. Jangjeon Math. Soc. **17** (2014), no. 4, 519–525. MR 3243798

[4] D. S. Kim and T. Kim, *Symmetry identities of q-Bernoulli polynomials of the second kind*, Indian J. Pure Appl. Math. **46** (2015), 85–90.

[5] D. S. Kim, T. Kim, S.-H. Lee, and J.-J. Seo, *Identities of symmetry for generalized q-Euler polynomials*, Int. J. Math. Anal. (Ruse) **8** (2014), no. 13-16, 619–626. MR 3205513. http://dx.doi.org/10.12988/ijma.2014.4378

[6] D. S. Kim, T. Kim, and J.-J. Seo, *Identities of symmetry for expansions of q-Euler polynomials*, Int. J. Math. Anal. (Ruse) **8** (2014), no. 5-8, 401–407. MR 3188619. http://dx.doi.org/10.12988/ijma.2014.4242

[7] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on* $\mathbb{Z}_p$, Russ. J. Math. Phys. **16** (2009), no. 1, 93–96. MR 2486809 (2010c:11028). http://dx.doi.org/10.1134/s1061920809010063

[8] J. J. Seo, D. V. Dolgy, Y. S. Jang, and T. Kim, *A note on symmetric identities of the generalized Carlitz q-Euler polynomials of the first kind*, Adv. Stud. Contemp. Math. **24** (2014), no. 3, 295–300.

Received: February 17, 2015; Published: March 14, 2015