Some Identities of Symmetry for Generalized Carlitz-type $q$-Euler Polynomials under the Symmetric Group of Degree Four

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Abstract

In this paper, we give some identities of symmetry for the generalized Carlitz-type $q$-Euler polynomials under symmetric group of degree four which are derived from the fermionic $p$-adic $q$-integrals on $\mathbb{Z}_p$.

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1. Introduction

Let \( p \) be a fixed prime number. Throughout this paper, \( \mathbb{Z}_p, \mathbb{Q}_p \) and \( \mathbb{C}_p \) will denote the ring of \( p \)-adic integers, the field of \( p \)-adic numbers and the completion of algebraic closure of \( \mathbb{Q}_p \). Let \( |\cdot|_p \) be the normalized \( p \)-adic absolute value with \( |p|_p = \frac{1}{p} \), and let \( q \) be an indeterminate in \( \mathbb{C}_p \) such that \( |1 - q|_p < \frac{1}{p - 1} \). The \( q \)-number of \( x \) is defined as

\[
[x]_q = \frac{1 - q^x}{1 - q}.
\]

Note that \( \lim_{q \to 1} [x]_q = x \).

Let \( \mathbb{C}((\mathbb{Z}_p)) \) be the space of continuous functions on \( \mathbb{Z}_p \). For \( f \in \mathbb{C}((\mathbb{Z}_p)) \), the fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) is defined by Kim to be

\[
I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_{-q}(x)
\]

\[
= \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) (-q)^x
\]

\[
= \lim_{N \to \infty} \frac{1 + q}{1 + q p^N} \sum_{x=0}^{p^N-1} f(x) (-q)^x,
\]

(see \[3, 4, 6, 7\]).

For \( d \in \mathbb{N} \) with \( (d,p) = 1 \) and \( d \equiv 1 \pmod{2} \), we set

\[
X = \lim_{N \to \infty} \mathbb{Z}/dp^N \mathbb{Z}, \quad X^* = \bigcup_{0 < a < dp, (a,p)=1} (a + dp \mathbb{Z}_p),
\]

and

\[
a + dp^N \mathbb{Z}_p = \{ x \in X | x \equiv a \pmod{dp} \}, \quad a \in \mathbb{Z}
\]

where \( 0 \leq a < dp \).

For \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \), let \( \chi \) be a Dirichlet character with conductor \( d \). It is well known that the generalized Euler polynomial attached to \( \chi \) are defined by the generating function to be

\[
\sum_{n=0}^{\infty} E_{n,\chi} (x) \frac{t^n}{n!} = \frac{2}{e^{at} + 1} \sum_{a=0}^{d-1} \chi(a) (-1)^a e^{at} e^{xt}, \quad (\text{see } [1-7]).
\]

(1.2)

When \( x = 0 \), \( E_{n,\chi} = E_{n,\chi}(0) \) are called the Euler numbers attached to \( \chi \). If \( \chi = \chi^0 \) is the trivial character, then \( E_n(x) = E_{n,\chi^0}(x) \) are called the Euler polynomials.

In \[8\], the generalized Carlitz-type \( q \)-Euler polynomials attached to \( \chi \) are defined as

\[
E_{n,\chi,q}(x) = \int_X \chi(y) [x + y]^n_q \, d\mu_{-q}(y), \quad (n \geq 0).
\]

(1.3)

Note that

\[
\lim_{q \to 1} E_{n,\chi,q}(x) = \lim_{q \to 1} \int_X \chi(y) [x + y]^n_q \, d\mu_{-q}(y)
\]
= \int_X \chi(y) (x + y)^n d\mu_{-1}(y) = E_{n,x}(x).

When \( x = 0 \), \( E_{n,x,q} = E_{n,x,q}(0) \) are called the generalized Carlitz-type \( q \)-Euler numbers attached to \( \chi \).

The purpose of this paper is to investigate symmetric properties and identities for the generalized Carlitz-type \( q \)-Euler polynomials attached to \( \chi \) under the symmetric group of degree four arising from fermionic \( p \)-adic \( q \)-integrals on \( \mathbb{Z}_{p} \).

2. Symmetry identities of the generalized Carlitz-type \( q \)-Euler polynomials attached to \( \chi \) under \( S_4 \)

Let \( w_1, w_2, w_3, w_4 \in \mathbb{N} \) with \( w_1 \equiv 1 \) (mod 2), \( w_2 \equiv 1 \) (mod 2), \( w_3 \equiv 1 \) (mod 2), \( w_4 \equiv 1 \) (mod 2). For \( d \in \mathbb{N} \) with \( d \equiv 1 \) (mod 2), let \( \chi \) be a Dirichlet character with conductor \( d \). From (1.1) and (1.3),

\[
\int_X \chi(y) e^{[w_1 w_2 w_3 y + w_1 w_2 w_3 w_4 x + w_4 w_3 w_2 y + w_4 w_2 w_3 y + w_3 x i + w_1 w_3 j + w_4 w_1 w_2 k]q^t} d\mu_{-q^{w_1 w_2 w_3}}(y) \tag{2.1}
\]

By (2.1), we get

\[
\frac{1}{[2]q^{w_1 w_2 w_3}} \sum_{i=0}^{d w_2 - 1} \sum_{j=0}^{d w_3 - 1} \sum_{k=0}^{d w_3 - 1} (-1)^{i+j+k} q^{w_3 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_2 w_3 k} \chi(i) \chi(j) \chi(k) \tag{2.2}
\]

\[
\times \int_X \chi(y) e^{[w_1 w_2 w_3 y + w_1 w_2 w_3 w_4 x + w_4 w_2 w_3 y + w_4 w_3 w_2 x + w_3 x i + w_2 x j + w_4 w_1 w_2 k]q^t} d\mu_{-q^{w_1 w_2 w_3}}(y)
\]

\[
= \lim_{N \to \infty} \frac{1}{1 + q^{d w_1 w_2 w_3 w_4}} \sum_{i=0}^{d w_2 - 1} \sum_{j=0}^{d w_3 - 1} \sum_{k=0}^{d w_4 - 1} (-1)^{i+j+k+l} q^{w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_3 w_2 k + w_1 w_2 w_3 l} \chi(ijk)
\]

\[
\times \sum_{y=0}^{p^N-1} (-q) d^{w_1 w_2 w_3 w_4} \chi(y) e^{[w_1 w_2 w_3 (l + w_4 y) + w_1 w_2 w_3 w_4 x + w_4 w_2 w_3 y + w_4 w_3 w_2 x + w_3 x i + w_2 x j + w_4 w_1 w_2 k]q^t}.
\]

Note that the equation (2.2) is invariant for any permutation \( \sigma \in S_4 \). Thus, we have the following theorem.
Theorem 2.1. Let $w_1, w_2, w_3, w_4 \in \mathbb{N}$, with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, $w_3 \equiv 1 \pmod{2}$, $w_4 \equiv 1 \pmod{2}$. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, let $\chi$ be a Dirichlet character with conductor $d$. Then, the following expressions

\[ \frac{1}{[2]_q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} (-1)^{i+j+k} \]

\[ \times q^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k} \chi(i) \chi(j) \chi(k) \]

\[ \times \int_X \chi(y) e^{[w_1 w_2 w_3 y + w_1 w_2 w_3 x + w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k]t} \, d\mu_{-q^{w_1 w_2 w_3}}(y) \]  

(2.3)

are the same for any $\sigma \in S_4$.

It is not difficult to show that

\[ \int_X \chi(y) [w_1 w_2 w_3 y + w_4 w_2 w_3 x + w_4 w_1 w_3 j + w_4 w_1 w_2 k]^n \, d\mu_{-q^{w_1 w_2 w_3}}(y) \]  

(2.4)

Thus, by (2.3), we get

\[ \int_X \chi(y) [w_1 w_2 w_3 y + w_4 w_2 w_3 x + w_4 w_1 w_3 j + w_4 w_1 w_2 k]^n \, d\mu_{-q^{w_1 w_2 w_3}}(y) \]  

(2.4)

are the same for any $\sigma \in S_4$.

Theorem 2.2. Let $n \geq 0$, $w_1, w_2, w_3, w_4, d \in \mathbb{N}$, with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, $w_3 \equiv 1 \pmod{2}$, $w_4 \equiv 1 \pmod{2}$, $d \equiv 1 \pmod{2}$, and let $\chi$ be a Dirichlet character with conductor $d$. Then, the following expressions

\[ \left[ \frac{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}{[2]_q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}} \right]^n \]

\[ \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} (-1)^{i+j+k} q^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k} \]

\[ \times \chi(i) \chi(j) \chi(k) E_{n, \chi, q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}} \left( w_{\sigma(4)}x + \frac{w_{\sigma(4)}}{w_{\sigma(1)}} i + \frac{w_{\sigma(4)}}{w_{\sigma(2)}} j + \frac{w_{\sigma(4)}}{w_{\sigma(3)}} k \right) \]

are the same for any $\sigma \in S_4$. 

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We observe that
\[
\begin{align*}
\left[ y + w_4x + \frac{w_4}{w_1}i + \frac{w_4}{w_2}j + \frac{w_4}{w_3}k \right]^n & = \sum_{m=0}^{n} \binom{n}{m} \left( \frac{[w_4]_q}{[w_1 w_2 w_3]_q} \right)^{n-m} \left[ w_2 w_3 i + w_1 w_3 j + w_1 w_2 k \right]_{q^{w_4}} \\
& \times q^{mw_2 w_3 i + w_1 w_3 j + w_1 w_2 w_3 k} \\
& \times [y + w_4x]_{q^{w_1 w_2 w_3}}.
\end{align*}
\]

Thus, by (1.3) and (2.5), we get
\[
\begin{align*}
\frac{[w_1 w_2 w_3]_q^n}{[2]_{q^{w_1 w_2 w_3}}} & \sum_{i=0}^{d_{w_1 - 1}} \sum_{j=0}^{d_{w_2 - 1}} \sum_{k=0}^{d_{w_3 - 1}} (-1)^{i+j+k} q^{w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k} \chi(i) \chi(j) \chi(k) \\
& \times \int_{\mathbb{F}_p} \left[ y + w_4 x + \frac{w_4}{w_1}i + \frac{w_4}{w_2}j + \frac{w_4}{w_3}k \right]^n \chi(y) d\mu_{-w_1 w_2 w_3}(y) \\
& = \sum_{m=0}^{n} \binom{n}{m} \frac{[w_1 w_2 w_3]_q^m}{[2]_{q^{w_1 w_2 w_3}}} [w_4]_q^{n-m} E_{m,\chi,q}^{w_1 w_2 w_3} (w_4 x) \hat{T}_{w_1,\chi,q}^{w_4} (w_1, w_2, w_3 | m),
\end{align*}
\]

where
\[
\hat{T}_{n,\chi,q} (w_1, w_2, w_3 | m)
\]
\[
= \sum_{i=0}^{d_{w_1 - 1}} \sum_{j=0}^{d_{w_2 - 1}} \sum_{k=0}^{d_{w_3 - 1}} (-1)^{i+j+k} \chi(ij,k) q^{(m+1)(w_2 w_3 i + w_1 w_3 j + w_1 w_2 k)} \\
\times [w_2 w_3 i + w_1 w_3 j + w_1 w_2 k]_{q^{n-m}}.
\]

Therefore, by (2.7), we obtain the following theorem.

**Theorem 2.3.** Let \( n \geq 0 \), \( w_1, w_2, w_3, w_4, d \in \mathbb{N} \) with \( w_1 \equiv 1 \pmod{2} \), \( w_2 \equiv 1 \pmod{2} \), \( w_3 \equiv 1 \pmod{2} \), \( w_4 \equiv 1 \pmod{2} \), \( d \equiv 1 \pmod{2} \), and let \( \chi \) be a Dirichlet character with conductor \( d \). Then, the following expressions
\[
\sum_{m=0}^{n} \binom{n}{m} \frac{[w_4(1)w_4(2)w_4(3)]_q^m}{[2]_{q^{w_4(1)w_4(2)w_4(3)}}} [w_4(4)]_q^{n-m} E_{m,\chi,q}^{w_4(1)w_4(2)w_4(3)} (w_4 x) \hat{T}_{n,\chi,q}^{w_4(4)} (w_4(1)w_4(2)w_4(3) | m)
\]
are the same for any \( \sigma \in S_4 \).

**References**


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