On Mixed Fractional Integrodifferential Equations with Fractional Non-separated Boundary Conditions

V. V. Kharat
Department of Mathematics,
N.K. Orchid College of Engg. and Tech., Solapur-413002, India (M.S.)

D. R. Hasabe
Department of Mathematics,
Y.C. Institue of Science Satara, Satara-415001, India (M.S.)

Abstract
The aim of the present paper is to establish the existence, uniqueness and boundedness of solutions of nonlinear mixed fractional integrodifferential equation with fractional non-separated boundary conditions in Banach spaces.

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1 Introduction
Let \( C(J, R) \) be the Banach space of all continuous functions from \( J = [0, T] \) into \( R \) endowed with the norm
\[
\| x \| = \sup \{ \| x(t) \| : t \in J \}.
\]
We consider the class of nonlinear fractional integrodifferential equations with fractional non-separated boundary value condition of the type
\[
_{C}D_{0}^{q} x(t) = f \left( t, x(t), \int_{0}^{t} k(t, s)x(s) \, ds, \int_{0}^{T} h(t, s)x(s) \, ds \right), \quad 1 < q \leq 2,
\]
\[ a_1 x(0) + b_1 x(T) = c_1, \]  
(1)  
\[ a_2 [C^\gamma x(0)] + b_2 [C^\gamma x(T)] = c_2, \quad 0 < \gamma < 1, \]  
(3)  

where \( C^\gamma \) is the Caputo fractional derivative of order \( q \), \( f : J \times R \times R \times R \to R \) is a continuous function, \( a_i, b_i, c_i, i = 1, 2 \) are real constants such that \( a_1 + b_1 \neq 0 \) and \( b_2 \neq 0 \).

Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economy and science. Characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integer-order models. The investigation of the theory of fractional differential and integral equations has started quite recently. One can see the monographs of Kilbas et al. [11], Podlubny [16]. Most of the practical systems are integrodifferential equations in nature and hence the study of integrodifferential systems is very important. Many forms of these equations are possible see [1] and the references therein.

Many recent papers have dealt with the existence, uniqueness and other properties of solutions of special forms of the equations (1) - (3), see [2, 3, 4, 5, 7, 8, 9, 12, 13, 14] and some of the references cited therein. Recently, S. D. Kendre et al. [10] investigated the existence and uniqueness of solutions of special form of (1) - (3). The aim of the present paper is to prove the existence, uniqueness and boundedness of solution of nonlinear fractional mixed integrodifferential equations (1) - (3). The main tools employed in our analysis are based on the theory of fractional calculus and fixed point theorems.

The paper is organized as follows: Section 2, presents the preliminaries. Section 3, deals with the main results. Finally, in section 4, we discuss an example to illustrate the theory.

## 2 Preliminaries

Before proceeding to the statement of our main results, we set forth definitions, preliminaries and hypotheses that will be used in our subsequent discussion. For more details see [6, 11, 17].

**Definition 1** The Riemann-Liouville fractional integral of order \( q \), is defined by  
\[ I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad q > 0, \]  
(4)  
provided the right hand side is pointwise defined on \((0, \infty)\).

**Definition 2** The Riemann-Liouville fractional derivative of order \( q > 0 \), for a continuous function \( f : (0, \infty) \to R \) is defined by  
\[ D^q f(t) = \frac{1}{\Gamma(n-q)} \left( \frac{d}{dt} \right)^n \int_0^t \frac{f(s)}{(t-s)^{q+1-n}} ds, \quad n = [q] + 1, \]  
(5)  
provided the right hand side is pointwise defined on \((0, \infty)\).
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Definition 3 For an at least \( n \)-times differentiable function \( f : [0, \infty) \rightarrow R \), the Caputo derivative of fractional of order \( q \) is defined by

\[
CD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1+q-n}} ds, \quad n - 1 < q < n, \quad n = [q] + 1, \tag{6}
\]

where \([q]\) denotes the integer part of the real number \( q \).

Lemma 2.1 Let \( q > 0 \), then the differential equation \( CD^q f(t) = 0 \) has solutions

\[
f(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},
\]

where \( c_i \in R, i = 0, 1, 2, \ldots, n-1; n = [q] + 1 \).

Lemma 2.2 Let \( q > 0 \), then

\[
I^q (CD^q f) (t) = f(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},
\]

where \( c_i \in R, i = 0, 1, 2, \ldots, n-1; n = [q] + 1 \).

Definition 4 A function \( x \in C^1(J, R) \) is said to be a solution of the boundary value problem (1)-(3), if \( x \) satisfies the equation

\[
CD^q x(t) = f \left( t, x(t), \int_0^t k(t,s)x(s) ds, \int_0^T h(t,s)x(s) ds \right) \text{ a.e. on } J, \tag{7}
\]

and the condition \( a_1 x(0) + b_1 x(T) = c_1, \quad a_2 [CD^\gamma x(0)] + b_2 [CD^\gamma x(T)] = c_2, \quad 0 < \gamma < 1 \).

We need the following results in our subsequent discussion.

Lemma 2.3 \cite{12} For any \( y \in C(J, R) \), the unique solution of the fractional non-separated boundary-value problem

\[
CD^q x(t) = y(t), \quad 1 < q \leq 2,
\]

\[
a_1 x(0) + b_1 x(T) = c_1,
\]

\[
a_2 [CD^\gamma x(0)] + b_2 [CD^\gamma x(T)] = c_2, \quad 0 < \gamma < 1,
\]

is given by

\[
x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s) ds - \frac{t \Gamma(2-\gamma)}{T^{1-\gamma}} \int_0^T (T-s)^{q-\gamma-1} y(s) ds + \frac{t \Gamma(2-\gamma)}{T^{1-\gamma} b_2} \int_0^T (T-s)^{q-\gamma-1} y(s) ds \left\{ \int_0^T (T-s)^{q-1} y(s) ds - T^{\gamma} \Gamma(2-\gamma) \int_0^T \frac{T^{\gamma} (T-s)^{q-\gamma-1} y(s) ds}{\Gamma(q-\gamma)} \right\} \nonumber
\]

\[
- \frac{b_1}{a_1 + b_1} \left( \frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{b_2} - c_1 \right).
\]

Lemma 2.4 (Schauder fixed point theorem) Let \( E \) be a Banach space, \( C \) a closed, convex and nonempty subset of \( E \), let \( F : C \rightarrow C \) be a continuous mapping such that \( F(C) \) is a relatively compact subset of \( E \). Then \( F \) has at least one fixed point in \( C \).

Lemma 2.5 \cite{6} (Nonlinear alternative of Leray-Schauder type) Let \( E \) be a Banach space, \( C \) a closed, convex subset of \( E \), \( U \) an open subset of \( C \) and \( 0 \in U \). Suppose that \( F : \bar{U} \rightarrow C \) is a continuous, compact (that is, \( F(\bar{U}) \) is a relatively compact subset of \( C \)) map. Then either:

(i) \( F \) has a fixed point in \( \bar{U} \); or

(ii) There is a \( u \in \partial U \) (the boundary of \( U \) in \( C \)) and \( \lambda \in (0,1) \) with \( u = \lambda F(u) \).
3 Main Results

Theorem 3.1 Suppose that $f$ satisfies

$$
\|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\| \leq m(t) \|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|
$$

for $t \in J$, $x_1$, $x_2$, $y_1$, $y_2$, $z_1$, $z_2 \in R$ with $m \in L^\infty(J, R^+)$. If

$$
\|m\|_{L^\infty(1 + T k_T + T h_T) T^q} \left(1 + \frac{|b_1|}{a_1 + b_1}\right) \left(\frac{1}{\Gamma(q + 1)} + \frac{\Gamma(2 - \gamma)}{\Gamma(q - \gamma + 1)}\right) < 1
$$

then the system (1) - (3) has a unique solution.

Proof: In view to apply contraction principle, we define an operator $F: C(J, R) \rightarrow C(J, R)$ by

$$(Fx)(t) = \frac{1}{\Gamma(q)} \int_0^1 (t - \tau)^{q-1} f \left(s, x(s), \int_0^s k(s, \tau)x(\tau) d\tau, \int_0^T h(s, \tau)x(\tau) d\tau\right) ds - \frac{t \Gamma(2 - \gamma)}{T^{1-\gamma}} \int_0^T (s - \tau)^{q-\gamma-1} f \left(s, x(s), \int_0^s k(s, \tau)x(\tau) d\tau, \int_0^T h(s, \tau)x(\tau) d\tau\right) ds
$$

$$
+ \frac{t \Gamma(2 - \gamma)c_2}{T^{1-\gamma} b_2} - \frac{b_1}{a_1 + b_1} \left\{ \int_0^T \frac{(s - \tau)^{q-1}}{\Gamma(q)} f \left(s, x(s), \int_0^s k(s, \tau)x(\tau) d\tau, \int_0^T h(s, \tau)x(\tau) d\tau\right) ds \right\}
$$

$$
-T \gamma \Gamma(2 - \gamma) \int_0^T \frac{(s - \tau)^{q-\gamma-1}}{\Gamma(q - \gamma)} f \left(s, x(s), \int_0^s k(s, \tau)x(\tau) d\tau, \int_0^T h(s, \tau)x(\tau) d\tau\right) ds
$$

$$
- \frac{1}{a_1 + b_1} \left(\frac{b_1 c_2 T^\gamma \Gamma(2 - \gamma)}{b_2} - c_1\right).
$$

Problem (1) - (3) has solution if and only if the operator $Fx$ has fixed points. Define two operators $F_1$ and $F_2$ as $Fx = F_1 x + F_2 x$ where

$$(F_1 x)(t) = \frac{1}{\Gamma(q)} \int_0^1 (t - \tau)^{q-1} f \left(s, x(s), \int_0^s k(s, \tau)x(\tau) d\tau, \int_0^T h(s, \tau)x(\tau) d\tau\right) ds,
$$

$$(F_2 x)(t) = -k_2^x t - k_1^x.
$$

Here $k_1^x$ and $k_2^x$ are constants given by

$$
k_1^x = \frac{b_1 c_2 T^\gamma \Gamma(2 - \gamma)}{(a_1 + b_1) b_2} - \frac{c_1}{a_1 + b_1}
$$

$$
+ \frac{b_1}{a_1 + b_1} \left\{ \int_0^T \frac{(s - \tau)^{q-1}}{\Gamma(q)} f \left(s, x(s), \int_0^s k(s, \tau)x(\tau) d\tau, \int_0^T h(s, \tau)x(\tau) d\tau\right) ds \right\}
$$

$$
- T \gamma \Gamma(2 - \gamma) \int_0^T \frac{(s - \tau)^{q-\gamma-1}}{\Gamma(q - \gamma)} f \left(s, x(s), \int_0^s k(s, \tau)x(\tau) d\tau, \int_0^T h(s, \tau)x(\tau) d\tau\right) ds
$$

$$
- \frac{1}{a_1 + b_1} \left(\frac{b_1 c_2 T^\gamma \Gamma(2 - \gamma)}{b_2} - c_1\right).
$$

$$
k_2^x = \frac{\Gamma(2 - \gamma)}{T^{1-\gamma}} \left\{ \int_0^T \frac{(s - \tau)^{q-\gamma-1}}{\Gamma(q - \gamma)} f \left(s, x(s), \int_0^s k(s, \tau)x(\tau) d\tau, \int_0^T h(s, \tau)x(\tau) d\tau\right) ds - \frac{c_2}{b_2}\right\}.
$$
For \( x, y \in C \) and for each \( t \in J \), we have
\[
\|(F_1x)(t) - (F_1y)(t)\| \\
\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|m(t)\| \left[ \|x(s) - y(s)\| + \int_0^s |k(s,\tau)||x(\tau) - y(\tau)||d\tau \\
+ \int_0^T |h(s,\tau)||x(\tau) - y(\tau)|| d\tau \right] ds \\
\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|m\|_L^\infty \left[ \|x - y\| + \int_0^s k_T\|x - y\||d\tau + \int_0^T h_T\|x - y\||d\tau \right] ds \\
\leq \|m\|_L^\infty \left[\|x - y\| + T k_T\|x - y\| + T h_T\|x - y\|\right] \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds \\
\leq \|m\|_L^\infty [1 + T k_T + T h_T] \frac{T^q}{\Gamma(q+1)} \|x - y\|, \quad (12)
\]
and
\[
\|(F_2x)(t) - (F_2y)(t)\| \leq T\|k^x_2 - k^y_2\| + \|k^x_1 - k^y_1\|, \quad (13)
\]
where
\[
T\|k^x_2 - k^y_2\| \leq T^\gamma \Gamma(2 - \gamma) \int_0^T \frac{(T-s)^{q-\gamma-1}}{\Gamma(q-\gamma)} \left\| f \left( s, x(s), \int_0^s k(s,\tau)x(\tau)\,d\tau, \int_0^T h(s,\tau)x(\tau)\,d\tau \right) - f \left( s, y(s), \int_0^s k(s,\tau)y(\tau)\,d\tau, \int_0^T h(s,\tau)y(\tau)\,d\tau \right) \right\| ds \\
\leq \|m\|_L^\infty [1 + T k_T + T h_T] \frac{\Gamma(2 - \gamma) T^q}{\Gamma(q - \gamma + 1)} \|x - y\|, \quad (14)
\]
and
\[
\|k^x_1 - k^y_1\| \leq \frac{|b_1|}{|a_1 + b_1|} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \|m\|_L^\infty [1 + T k_T + T h_T] \|x - y\| ds \\
+ \frac{|b_1| \Gamma(2 - \gamma)}{|a_1 + b_1|} \int_0^T \frac{(T-s)^{q-\gamma-1}}{\Gamma(q-\gamma)} \|m\|_L^\infty [1 + T k_T + T h_T] \|x - y\| ds \\
\leq \|m\|_L^\infty [1 + T k_T + T h_T] \|x - y\| \frac{|b_1|}{|a_1 + b_1|} \left[ \frac{T^q}{\Gamma(q+1)} + \frac{\Gamma(2 - \gamma) T^q}{\Gamma(q - \gamma + 1)} \right]. \quad (15)
\]
Therefore, from (12)-(15) we obtain
\[
\|F x - F y\| \leq \|m\|_L^\infty (1 + T k_T + T h_T) T^q \left( 1 + \frac{|b_1|}{|a_1 + b_1|} \right) \left( \frac{1}{\Gamma(q+1)} + \frac{\Gamma(2 - \gamma)}{\Gamma(q - \gamma + 1)} \right) \|x - y\|
\]
From (8) this shows that the map \( F \) is a contraction mapping. So, by the contraction mapping principle, \( F \) has a unique fixed point \( x \). That is, \( x \) is the unique solution of (1) - (3).
Corollary 3.2 Suppose that f satisfies
\[ \|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\| \leq L[\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|] \]
for \( t \in J, x_1, x_2, y_1, y_2, z_1, z_2 \in R \) and \( L > 0 \). If
\[ L(1 + T^q_1 + T^q_2)T^q \left( 1 + \frac{|b_1|}{|a_1 + b_1|} \right) \left( \frac{1}{\Gamma(q + 1)} + \frac{\Gamma(2 - \gamma)}{\Gamma(q - \gamma + 1)} \right) < 1 \] (16)
then the system (1) - (3) has a unique solution.

Theorem 3.3 Assume that
\[ \|f(t, x, y, z)\| \leq m(t) + d\|x\|^\rho + e\|y\|^\rho + g\|z\|^\rho \]
for \( t \in J, x, y, z \in R \) with \( m \in L^\infty(J, R^+), d, e, g \geq 0 \) and \( 0 \leq \rho < 1 \). Then the boundary value problem (1) - (3) has at least one solution on \( J \).

Proof: Define \( B_r = \{ x \in C(J, R) : \|x\| \leq r \} \), where \( r = \max \left\{ 2A, (2Bn)^{\frac{1}{1-\rho}} \right\} \),
\[ n = d + ekT^\rho T^\rho + ghT^\rho T^\rho, \]
\[ A = \frac{T^\gamma \Gamma(2 - \gamma)|c_2|}{|b_2|} + \frac{b_1c_1T^\gamma \Gamma(2 - \gamma)}{(a_1 + b_1)b_2} - \frac{c_1}{a_1 + b_1} \times \]
\[ \|m\|_{L^\infty}T^q \left( 1 + \frac{|b_1|}{|a_1 + b_1|} \right) \left( \frac{1}{\Gamma(q + 1)} + \frac{\Gamma(2 - \gamma)}{\Gamma(q - \gamma + 1)} \right) \] (17)
and
\[ B = T^q \left( 1 + \frac{|b_1|}{|a_1 + b_1|} \right) \left( \frac{1}{\Gamma(q + 1)} + \frac{\Gamma(2 - \gamma)}{\Gamma(q - \gamma + 1)} \right). \] (18)
It is obvious that \( B_r \) is a closed, bounded and convex subset of the Banach space \( C(J, R) \). Firstly, we prove that \( F : B_r \rightarrow B_r \), where \( Fx = F_1x + F_2x \) and \( F_1, F_2 \) are as defined by (10), (11) respectively. For any \( x \in B_r \), we have
\[ \|(F_1x)(t)\| \]
\[ \leq \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \left[ m(s) + d\|x(s)\|^\rho + e \left| \int_0^s k(s, \tau)x(\tau)\,d\tau \right| + g \left| \int_0^T h(s, \tau)x(\tau)\,d\tau \right|^\rho \right] ds \]
\[ \leq \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} (\|m\|_{L^\infty} + (d + ekT^\rho T^\rho + ghT^\rho T^\rho)\|x(s)\|^\rho) \, ds \]
\[ \leq \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} (\|m\|_{L^\infty} + nr^\rho) \, ds \]
\[ \leq \frac{\|m\|_{L^\infty} + nr^\rho}{\Gamma(q + 1)} \frac{T^q}{\Gamma(q)} \] (19)
and
\[ \|(F_2x)(t)\| = \| - k^x_2t - k^x_1 \| \leq t\|k^x_2\| + \|k^x_1\| \leq T\|k^x_2\| + \|k^x_1\|, \] (20)
where

\[ T\|k_2\| \leq T^{\gamma}\Gamma(2 - \gamma) \int_0^T \frac{(T - s)^{q-\gamma-1}}{\Gamma(q-\gamma)} \left\| f \left(s, x(s), \int_0^s k(s, \tau)x(\tau) \, d\tau, \int_0^T h(s, \tau)x(\tau) \, d\tau \right) \right\| ds \\
+ \frac{T^{\gamma}\Gamma(2 - \gamma)|c_2|}{|b_2|} \leq \left[ \|m\|_{L^\infty} + nr^q \right] \frac{T^{\gamma}\Gamma(2 - \gamma)}{\Gamma(q - \gamma + 1)} + \frac{T^{\gamma}\Gamma(2 - \gamma)|c_2|}{|b_2|} \] (21)

and

\[ \|k_1\| \leq \left\| \frac{b_1c_2 T^{\gamma}\Gamma(2 - \gamma)}{(a_1 + b_1)b_2} - \frac{c_1}{a_1 + b_1} \right\| + \left\{ \int_0^T \frac{(T - s)^{q-1}}{\Gamma q} \left\| f \left(s, x(s), \int_0^s k(s, \tau)x(\tau) \, d\tau, \int_0^T h(s, \tau)x(\tau) \, d\tau \right) \right\| ds + \frac{T^{\gamma}\Gamma(2 - \gamma)}{\Gamma(q - \gamma)} \right\} \] (22)

Hence, from (19)-(22) we obtain

\[ \|Fx\| \leq \frac{T^{\gamma}\Gamma(2 - \gamma)|c_2|}{|b_2|} + \left\| \frac{b_1c_2 T^{\gamma}\Gamma(2 - \gamma)}{(a_1 + b_1)b_2} - \frac{c_1}{a_1 + b_1} \right\| + \|m\|_{L^\infty} T^q \left( 1 + \frac{|b_1|}{|a_1 + b_1|} \right) \left( \frac{1}{\Gamma(q + 1)} + \frac{\Gamma(2 - \gamma)}{\Gamma(q - \gamma + 1)} \right) + nr^q T^q \left( 1 + \frac{|b_1|}{|a_1 + b_1|} \right) \left( \frac{1}{\Gamma(q + 1)} + \frac{\Gamma(2 - \gamma)}{\Gamma(q - \gamma + 1)} \right) \leq A + nr^q B \leq \frac{r}{2} + \frac{r}{2} \leq r. \]

This implies that \( F : B_r \to B_r \).

Secondly, we show that \( F \) maps bounded sets into equicontinuous sets. Let \( \tilde{B} \) be any bounded subset of \( C(J, R) \). Since \( f \) is continuous, we can assume without any loss of generality that there is positive constant \( N \) such that

\[ \|f \left(s, x(s), \int_0^s k(s, \tau)x(\tau) \, d\tau, \int_0^T h(s, \tau)x(\tau) \, d\tau \right) \| \leq N \]

for any \( t \in J \) and \( x \in \tilde{B} \). Now let \( 0 \leq t_1 < t_2 \leq T \). For each \( x \in \tilde{B} \), we have
\[
\| (F_1x)(t_2) - (F_1x)(t_1) \| \\
\leq \left\| \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} f \left( s, x(s), \int_0^s k(s, \tau)x(\tau) d\tau, \int_0^T h(s, \tau)x(\tau) d\tau \right) ds \right\| \\
+ \left\| \frac{1}{\Gamma(q)} \int_0^{t_1} \left[ (t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right] f \left( s, x(s), \int_0^s k(s, \tau)x(\tau) d\tau, \int_0^T h(s, \tau)x(\tau) d\tau \right) ds \right\|
\]
\[
\leq \frac{N(t_2 - t_1)^q}{\Gamma(q + 1)} + \frac{N[t_2^q - (t_2 - t_1)^q - t_1^q]}{\Gamma(q + 1)} \leq \frac{N[t_2^q - t_1^q]}{\Gamma(q + 1)}. \tag{23}
\]
From (23)-(24), we have
\[
\| (F_2x)(t_2) - (F_2x)(t_1) \| \leq \Gamma(2 - \gamma) \left( \frac{NT^{q-\gamma}}{\Gamma(q - \gamma + 1)} + \frac{|c_2|}{|b_2|} \right) (t_2 - t_1). \tag{24}
\]
as \( t_2 \to t_1 \) independently of \( x \in \bar{B} \). In view of the continuity of the function \( f \), it is clear that the operator \( F \) is continuous. From the above analysis and Arzela-Ascoli Theorem tells us that \( F(B_r) \) is a relatively compact subset of \( C(J,R) \). Thus by Schauder fixed point Theorem the problem (1) - (3) has at least one solution.

**Corollary 3.4** Assume that \( \| f(t,x,y,z) \| \leq v(t) \) for \( t \in J, x, y, z \in R \) with \( v \in C(J,R^+) \). Then the boundary value problem (1) - (3) has at least one solution.

**Corollary 3.5** Assume that \( \| f(t,x,y,z) \| \leq m(t) + d\|x\| + e\|y\| + g\|z\| \) for \( t \in J, x, y, z \in R \) with \( m \in L^\infty(J,R^+) \), \( d, e, g \geq 0 \). If \( nB < 1 \) \( [B \text{ is defined by (18)}] \), then the boundary value problem (1) - (3) has at least one solution on \( J \).

**Theorem 3.6** Assume that

1. there exists a function \( m \in L^\infty(J,R^+) \) and a non decreasing function \( \psi : [0, \infty) \to [0, \infty) \) such that \( \| f(t,x,y,z) \| \leq m(t)\psi(\|x\| + \|y\| + \|z\|) \) for \( t \in J, x, y, z \in R \).
2. there exists a constant \( M > 0 \) such that
   \[
   M > O + \psi[(1 + Tk_T + Th_T)M]Q \tag{25}
   \]
   where
   \[
   O = \frac{T^\gamma\Gamma(2 - \gamma)|c_2|}{|b_2|} + \left| \frac{b_1c_2T^\gamma\Gamma(2 - \gamma)}{(a_1 + b_1)b_2} - \frac{c_1}{a_1 + b_1} \right|
   \]
   and
   \[
   Q = \|m\|_{L^\infty}T^q \left( 1 + \frac{|b_1|}{a_1 + b_1} \right) \left( \frac{1}{\Gamma(q + 1)} + \frac{\Gamma(2 - \gamma)}{\Gamma(q - \gamma + 1)} \right).
   \]
Then problem (1) - (3) has at least one solution on $J$.

**Proof:** Let $F x = F_1 x + F_2 x$ where $F_1$ and $F_2$ are defined by (10) and (11). We first prove that $F$ maps bounded sets into bounded sets in $C(J,R)$. Let $B$ be a bounded subset of $C(J,R)$ and assume that $\|x\| \leq r$ for any $x \in B$. As proved in the above Theorem 3.3, we have the following estimates

$$
\|(F_1 x)(t)\| \leq \|m\|_{L_\infty} \psi[(1 + T k T + T h T)] \|x\| \int_0^t (t-s)^{\gamma-1} \frac{ds}{\Gamma(q)} \leq \|m\|_{L_\infty} \psi[(1 + T k T + T h T)] T^q \Gamma(q+1),
$$

and

$$
\|(F_2 x)(t)\| = \| - k_2^x t - k_1^x \| \leq T \|k_2^x\| + \|k_1^x\|,
$$

where

$$
T \|k_2^x\| \leq \|m\|_{L_\infty} \psi[(1 + T k T + T h T)] T^q \frac{\Gamma(2 - \gamma) T^q}{\Gamma(q+1)} + \frac{|c_2| \Gamma(2 - \gamma) T^q}{|b_2|}
$$

and

$$
\|k_1^x\| \leq \frac{|b_1|}{|a_1 + b_1|} \left\{ \|m\|_{L_\infty} \psi[(1 + T k T + T h T)] T^q \frac{\Gamma(2 - \gamma)}{\Gamma(q+1)} + \frac{|b_1 c_2 T^\gamma T^q}{(a_1 + b_1) b_2} - \frac{c_1}{a_1 + b_1} \right\}.
$$

Hence, we have

$$
\|F x\| \leq \|m\|_{L_\infty} \psi[(1 + T k T + T h T)] T^q \frac{\Gamma(2 - \gamma) T^q}{\Gamma(q+1)} + \|m\|_{L_\infty} \psi[(1 + T k T + T h T)] T^q \frac{\Gamma(2 - \gamma)}{\Gamma(q+1)} + \frac{|b_1|}{|a_1 + b_1|} \left\{ \|m\|_{L_\infty} \psi[(1 + T k T + T h T)] T^q \frac{\Gamma(2 - \gamma)}{\Gamma(q+1)} + \frac{|b_1 c_2 T^\gamma T^q}{(a_1 + b_1) b_2} - \frac{c_1}{a_1 + b_1} \right\}.
$$

This implies that $F(B)$ is bounded in $C(J,R)$.

Secondly, we claim that $F$ is equicontinuous on bounded subsets of $C(J,R)$. The proof of this claim is the same as the corresponding part of Theorem 3.3. Finally, let $x = \lambda F x$ for some $\lambda \in (0,1)$. Then for each $t \in J$, we have

$$
\|x(t)\| = \|x(F x)(t)\| \leq O + \psi[(1 + T k T + T h T)]\|x\|| Q.
$$
We consider the following nonlinear fractional integrodifferential equations with fractional non-

In this section we give the application of our main results established in previous section.

4 Application

In this section we give the application of our main results established in previous section. We consider the following nonlinear fractional integrodifferential equations with fractional non-separated boundary value condition

\[ \text{For } x \in J \text{ such that } x = \lambda Fx \text{ for some } \lambda \in (0, 1). \text{ As a consequence of Lemma 2.5, we deduce that } F \text{ has a fixed point } x \in \bar{U} \text{ which is a solution of the problem (1) - (3).} \]

\[ \text{Problem (26)-(28) is of the form (1) - (3) with } q = \frac{3}{2}, \gamma = \frac{1}{2}, a_1 = 3, b_1 = \frac{1}{2}, c_1 = 2.5, a_2 = 2, b_2 = \frac{1}{3}, c_2 = \frac{2}{3}, T = 1 \text{ and} \]

\[ f(t, x(t), K_1x(t), H_1x(t)) = \frac{1}{(t + 4)^2} \left[ \frac{|x(t)|}{1 + |x(t)|} + \sin^2 t \right] + \frac{1}{36} K_1x(t) + \frac{1}{32} H_1x(t), \]

where

\[ K_1x(t) = \int_0^t e^{-t}(4 + t)^2 x(s)ds, \quad H_1x(t) = \int_0^1 \frac{1}{(3 + t)^2} x(s)ds. \]

\[ \text{For } x_1, x_2 \in X \text{ and } t \in J, \text{ we have,} \]

\[ \|f(t, x_1, K_1x_1, H_1x_1) - f(t, x_2, K_1x_2, H_1x_2)\|
\]

\[ \leq \frac{1}{(t + 4)^2} \|x_1(t) - x_2(t)\| + \frac{1}{36} \|K_1x_1(t) - K_1x_2(t)\| + \frac{1}{32} \|H_1x_1(t) - H_1x_2(t)\|
\]

\[ \leq \frac{1}{16} \left[ \|x_1 - x_2\| + \|K_1x_1 - K_1x_2\| + \|H_1x_1 - H_1x_2\| \right]. \]

As \( k_T = \frac{1}{16}, \quad h_T = \frac{1}{9} \text{ and } m(t) = \frac{1}{16} \) we have

\[ \|m\|_{L^q(1 + Tk_T + Th_T^q)} \left( 1 + \frac{|b_1|}{|a_1 + b_1|} \right) \left( \frac{1}{\Gamma(q + 1)} + \frac{\Gamma(2 - \gamma)}{\Gamma(q - \gamma + 1)} \right) \]
\[
\begin{align*}
&= \frac{1}{16} \left[ 1 + \frac{1}{16} + \frac{1}{9} \right] \times \frac{8}{7} \times [0.7523 + 0.8862] \\
&= 0.13735 < 1
\end{align*}
\]

Thus, all the assumptions of the Theorem 3.1 are satisfied. Therefore problem (26)-(28) has a unique solution.

References


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