International Journal of Mathematical Analysis
Vol. 9, 2015, no. 13, 599 - 609
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijma.2015.511

# Positive Solutions and Iterative Approximations for a Second Order Difference Equation

**Qinhua Wu**

Department of Basic Science Subjects
Huaihai Institute of Technology
Lianyungang, Jiangsu 222005, P. R. China

**Chahn Yong Jung***

Department of Business Administration
Gyeongsang National University
Jinju 660-701, Korea
*Corresponding author

**Shin Min Kang**

Department of Mathematics and RINS
Gyeongsang National University
Jinju 660-701, Korea



**Abstract**

The existence of bounded positive solutions for a second order neutral difference equation and convergence of the Mann iterative methods relative to the bounded positive solutions are proved.

**Mathematics Subject Classification:** 39A10, 39A20, 39A22

**Keywords:** second order difference equation, bounded positive solution, Banach fixed point, Mann iterative method

# 1 Introduction and preliminaries

The oscillation and asymptotic behavior of second order neutral delay difference equations have been studied in [2, 3, 4]. Cheng [1] proved the existence of a nonoscillatory solution for the second order neutral delay difference equation with positive and negative coefficients

$$\Delta^2\big(x(n)+px(n-m)\big)+p(n)x(n-k)-q(n)x(n-l)=0, \quad n\geq n_0.$$

The aim of this paper is to study the following second order difference equation

$$\Delta^2\big(x(n)+b(n)x(n-\tau)\big)+f\big(n,x(n-c),x(n-d)\big)=a(n), \quad n\geq n_0, \quad (1.1)$$

where $\tau\in\mathbb{N}$, $n_0\in\mathbb{N}_0$, $c,d\in\mathbb{Z}$, $a,b:\mathbb{N}_{n_0}\to\mathbb{R}$ and $f:\mathbb{N}_{n_0}\times\mathbb{R}^2\to\mathbb{R}$. Using the Banach fixed point theorem, we show the existence of bounded positive solutions for Eq.(1.1) and convergence of the Mann iterative methods relative to the bounded positive solutions.

Throughout this paper, we assume that $\Delta$ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, $\mathbb{R}=(-\infty,+\infty)$, $\mathbb{R}^+=[0,+\infty)$, $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of all integers and positive integers, respectively,

$$\begin{aligned}\mathbb{Z}_t&=\{n:n\in\mathbb{Z}\text{ with }n\geq t\},\\ \mathbb{N}_t&=\{n:n\in\mathbb{N}\text{ with }n\geq t\}, \quad t\in\mathbb{Z},\\ \beta&=\inf\{n_0-\tau,n_0-c,n_0-d\},\end{aligned}$$

$l^\infty_\beta$ represents the Banach space of all bounded sequences on $\mathbb{Z}_\beta$ with norm

$$\|x\|=\sup_{n\in\mathbb{Z}_\beta}|x(n)| \text{ for } x=\{x(n)\}_{n\in\mathbb{Z}_\beta}\in l^\infty_\beta$$

and, for $M>N>0$

$$A(N,M)=\big\{x=\{x(n)\}_{n\in\mathbb{Z}_\beta}\in l^\infty_\beta : N\leq x(n)\leq M,\ n\in Z_\beta\big\}.$$

It is clear that $A(N,M)$ is a bounded closed subset of the Banach space $l^\infty_\beta$. By a solution of Eq.(1.1), we mean a sequence $\{x(n)\}_{n\in\mathbb{Z}_\beta}$ with a positive integer $T\geq n_0+\tau+|\beta|$ such that Eq.(1.1) is satisfied for all $n\geq T$.

# 2 Bounded positive solutions and convergence of the Mann iterative methods

Our main results are as follows.

**Theorem 2.1.** *Let $M$ and $N$ be two positive constants with $M > N$ and*

$$b(n) = -1, \quad \textit{eventually.} \tag{2.1}$$

*Suppose that there exist two mappings $P, Q : \mathbb{N}_{n_0} \to \mathbb{R}^+$ satisfying*

$$\begin{aligned} &|f(n, u_1, u_2) - f(n, \bar{u}_1, \bar{u}_2)| \le P(n) \max\{|u_1 - \bar{u}_1|, |u_2 - \bar{u}_2|\}, \\ &\quad n \in \mathbb{N}_{n_0},\ u_1, u_2, \bar{u}_1, \bar{u}_2 \in [N, M]; \end{aligned} \tag{2.2}$$

$$|f(n, u_1, u_2)| \le Q(n), \quad n \in \mathbb{N}_{n_0},\ u_1, u_2 \in [N, M]; \tag{2.3}$$

$$\sum_{i=1}^{\infty} \sum_{s=n_0+i\tau}^{\infty} \sum_{t=s}^{\infty} P(t) < +\infty; \tag{2.4}$$

$$\sum_{i=1}^{\infty} \sum_{s=n_0+i\tau}^{\infty} \sum_{t=s}^{\infty} [Q(t) + |a(t)|] < +\infty. \tag{2.5}$$

*Then there exist $\theta \in (0, 1)$ and $T \ge n_0 + \tau + |\beta|$ such that for each $x_0 \in A(N, M)$, the Mann iterative method $\{x_m\}_{m \ge 0}$ generated by*

$$x_{m+1}(n) = \begin{cases} (1 - \alpha_m) x_m(n) + \alpha_m \Big\{ \frac{N+M}{2} \\ \quad + \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} [f(t, x_m(t-c), x_m(t-d)) \\ \quad -a(t)] \Big\}, \quad n \ge T,\ m \ge 0, \\ (1 - \alpha_m) x_m(T) + \alpha_m \Big\{ \frac{N+M}{2} \\ \quad + \sum_{i=1}^{\infty} \sum_{s=T+i\tau}^{\infty} \sum_{t=s}^{\infty} [f(t, x_m(t-c), x_m(t-d)) \\ \quad -a(t)] \Big\}, \quad \beta \le n < T,\ m \ge 0 \end{cases} \tag{2.6}$$

*converges to a bounded positive solution $x \in A(N, M)$ of Eq.(1.1) and has the following error estimate:*

$$\|x_{m+1} - x\| \le e^{-(1-\theta) \sum_{k=0}^{m} \alpha_k} \|x_0 - x\|, \quad m \in \mathbb{N}_0, \tag{2.7}$$

*where $\{\alpha_m\}_{m \in \mathbb{N}_0}$ is a sequence in $[0, 1]$ with*

$$\sum_{m=0}^{\infty} \alpha_m = +\infty. \tag{2.8}$$

*Proof.* It follows from (2.1), (2.4) and (2.5) that there exist $\theta \in (0, 1)$ and $T \ge n_0 + \tau + |\beta|$ satisfying

$$b(n) = -1, \quad n \ge T; \tag{2.9}$$

$$\theta = \sum_{i=1}^{\infty} \sum_{s=T+i\tau}^{\infty} \sum_{t=s}^{\infty} P(t); \tag{2.10}$$

$$\sum_{i=1}^{\infty}\sum_{s=T+i\tau}^{\infty}\sum_{t=s}^{\infty}[Q(t)+|a(t)|]<\frac{M-N}{2}. \tag{2.11}$$

Define a mapping $S: A(N,M)\to l_{\beta}^{\infty}$ by

$$Sx(n)=\begin{cases}\frac{N+M}{2}+\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}[f(t,x(t-c),x(t-d))\\ -a(t)], \quad n\geq T,\ x\in A(N,M),\\ Sx(T), \quad \beta\leq n<T,\ x\in A(N,M).\end{cases} \tag{2.12}$$

In light of (2.2), (2.3) and (2.10)∼(2.12), we conclude that

$$\begin{aligned}
&|Sx(n)-Sy(n)|\\
&=\left|\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}[f(t,x(t-c),x(t-d))-f(t,y(t-c),y(t-d))]\right|\\
&\leq\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}|f(t,x(t-c),x(t-d))-f(t,y(t-c),y(t-d))|\\
&\leq\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}P(t)\max\{|x(t-c)-y(t-c)|,|x(t-d)-y(t-d)|\}\\
&\leq\sum_{i=1}^{\infty}\sum_{s=T+i\tau}^{\infty}\sum_{t=s}^{\infty}P(t)\|x-y\|\\
&=\theta\|x-y\|, \quad n\geq T,\ x,y\in A(N,M),
\end{aligned}$$

$$\begin{aligned}
Sx(n)&=\frac{N+M}{2}+\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}[f(t,x(t-c),x(t-d))-a(t)]\\
&\leq\frac{N+M}{2}+\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}[|f(t,x(t-c),x(t-d))|+|a(t)|]\\
&\leq\frac{N+M}{2}+\sum_{i=1}^{\infty}\sum_{s=T+i\tau}^{\infty}\sum_{t=s}^{\infty}[Q(t)+|a(t)|]\\
&\leq\frac{N+M}{2}+\frac{M-N}{2}\\
&=M, \quad n\geq T,\ x,y\in A(N,M)
\end{aligned}$$

and

$$\begin{aligned}
Sx(n) &= \frac{N+M}{2} + \sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}[f(t,x(t-c),x(t-d)) - a(t)] \\
&\geq \frac{N+M}{2} - \sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}[|f(t,x(t-c),x(t-d)))| + |a(t)|] \\
&\geq \frac{N+M}{2} - \sum_{i=1}^{\infty}\sum_{s=T+i\tau}^{\infty}\sum_{t=s}^{\infty}[Q(t) + |a(t)|] \\
&\geq \frac{N+M}{2} - \frac{M-N}{2} \\
&= N, \quad n \geq T,\ x, y \in A(N,M),
\end{aligned}$$

which yields that, for all $x, y \in A(N,M)$

$$S(A(N,M)) \subseteq A(N,M) \quad \text{and} \quad \|Sx - Sy\| \leq \theta\|x-y\|. \tag{2.13}$$

It follows from (2.13) that $S$ is a contraction mapping in the nonempty closed set $A(N,M)$ and it has a unique fixed point $x \in A(N,M)$, that is,

$$x(n) = \frac{N+M}{2} + \sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}[f(t,x(t-c),x(t-d)) - a(t)], \quad n \geq T$$

and

$$\begin{aligned}
x(n-\tau) = \frac{N+M}{2} &+ \sum_{i=1}^{\infty}\sum_{s=n+(i-1)\tau}^{\infty}\sum_{t=s}^{\infty}[f(t,x(t-c),x(t-d)) - a(t)], \\
&n \geq T+\tau,
\end{aligned}$$

which imply that

$$x(n) - x(n-\tau) = -\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}[f(t,x(t-c),x(t-d)) - a(t)], \quad n \geq T+\tau$$

and

$$\Delta(x(n) - x(n-\tau)) = \sum_{t=n}^{\infty}[f(t,x(t-c),x(t-d)) - a(t)], \quad n \geq T+\tau,$$

which gives that

$$\Delta^2(x(n) - x(n-\tau))\big) = -f(n,x(n-c),x(n-d)) + a(n), \quad n \geq T+\tau,$$

that is, $x$ a bounded positive solution of Eq.(1.1). It follow from (2.6), (2.10), (2.12) and (2.13) that

$$\begin{aligned}
&|x_{m+1}(n) - x(n)| \\
&= \Bigg|(1-\alpha_m)x_m(n) + \alpha_m\Bigg\{\frac{N+M}{2} \\
&\quad + \sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}[f(t, x_m(t-c), x_m(t-d)) - a(t)]\Bigg\} - x(n)\Bigg| \\
&\le (1-\alpha_m)|x_m(n) - x(n)| + \alpha_m|Sx_m(n) - Sx(n)| \\
&\le (1-\alpha_m)\|x_m - x\| + \alpha_m\theta\|x_m - x\| \\
&= (1-(1-\theta)\alpha_m)\|x_m - x\| \\
&\le e^{-(1-\theta)\alpha_m}\|x_m - x\|, \quad n \ge T,\ m \in \mathbb{N}_0,
\end{aligned}$$

which guarantees that

$$\begin{aligned}
\|x_{m+1} - x\| &\le e^{-(1-\theta)\alpha_m}\|x_m - x\| \\
&\le e^{-(1-\theta)\sum_{k=0}^{m}\alpha_k}\|x_0 - x\|, \quad m \in \mathbb{N}_0,
\end{aligned} \tag{2.14}$$

which together with (2.8) implies that $\lim_{m\to\infty} x_m = x$. This completes the proof. □

**Theorem 2.2.** *Assume that there exists a constant $\bar{b} \in [0,1)$ satisfying*

$$0 \le b(n) \le \bar{b}, \quad \textit{eventually.} \tag{2.15}$$

*Let $M$ and $N$ be two positive constants with $(1-\bar{b})M > N$ and there exist two mappings $P, Q : \mathbb{N}_{n_0} \to \mathbb{R}^+$ satisfying (2.2), (2.3),*

$$\sum_{s=n_0}^{\infty}\sum_{t=s}^{\infty} P(t) < +\infty \tag{2.16}$$

*and*

$$\sum_{s=n_0}^{\infty}\sum_{t=s}^{\infty} Q(t) < +\infty. \tag{2.17}$$

*Then there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + |\beta|$ such that for any $x_0 \in A(N,M)$, the Mann iterative method $\{x_m\}_{m\ge 0}$ generated by*

$$x_{m+1}(n) = \begin{cases}
(1-\alpha_m)x_m(n) + \alpha_m\big\{\frac{M(1+\bar{b})+N}{2} - b(n)x_m(n-\tau) \\
\quad + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[f(t, x_m(t-c), x_m(t-d)) - a(t)]\big\}, \\
\quad n \ge T+1, m \ge 0, \\
(1-\alpha_m)x_m(T+1) \\
\quad + \alpha_m\big\{\frac{M(1+\bar{b})+N}{2} - b(T+1)x_m(T+1-\tau) \\
\quad + \sum_{t=T}^{\infty}[f(t, x_m(t-c), x_m(t-d)) - a(t)]\big\}, \\
\quad \beta \le n \le T, m \ge 0
\end{cases} \tag{2.18}$$

*converges to a bounded positive solution* $x \in A(N, M)$ *of Eq.*(1.1) *and satisfies the error estimate* (2.7)*, where* $\{\alpha_m\}_{m\geq 0}$ *is a sequence in* $[0, 1]$ *satisfying* (2.8).

*Proof.* It follows from (2.16)∼(2.18) that there exist $\theta \in (0, 1)$ and $T \geq n_0 + \tau + |\beta|$ such that

$$0 \leq b(n) \leq \bar{b}, \quad n \geq T; \tag{2.19}$$

$$\theta = \bar{b} + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} P(t); \tag{2.20}$$

$$\sum_{s=T}^{\infty} \sum_{t=s}^{\infty} [Q(t) + |a(t)|] < \frac{M(1-\bar{b}) - N}{2}. \tag{2.21}$$

Define a mapping $S : A(N, M) \to l_\beta^\infty$ by

$$Sx(n) = \begin{cases} \frac{M(1+\bar{b})+N}{2} - b(n)x(n-\tau) \\ \quad + \sum_{s=T}^{n-1} \sum_{t=s}^{\infty} [f(t, x(t-c), x(t-d)) - a(t)], \\ \quad n \geq T+1,\ x \in A(N, M), \\ Sx(T+1), \quad \beta \leq n \leq T,\ x \in A(N, M). \end{cases} \tag{2.22}$$

By means of (2.2), (2.3) and (2.20)∼(2.22), we infer that

$$\begin{aligned} &|Sx(n) - Sy(n)| \\ &= \Bigg| \frac{M(1+\bar{b})+N}{2} - b(n)x(n-\tau) + \sum_{s=T}^{n-1} \sum_{t=s}^{\infty} [f(t, x(t-c), x(t-d)) - a(t)] \\ &\quad - \frac{M(1+\bar{b})+N}{2} + b(n)y(n-\tau) - \sum_{s=T}^{n-1} \sum_{t=s}^{\infty} [f(t, y(t-c), y(t-d)) - a(t)] \Bigg| \\ &\leq b(n)|x(n-\tau) - y(n-\tau)| \\ &\quad + \sum_{s=T}^{n-1} \sum_{t=s}^{\infty} |f(t, x(t-c), x(t-d)) - f(t, y(t-c), y(t-d))| \\ &\leq \bar{b}\|x-y\| + \sum_{s=T}^{n-1} \sum_{t=s}^{\infty} P(t) \max\{|x(t-c) - y(t-c)|, |x(t-d) - y(t-d)|\} \\ &\leq \left(\bar{b} + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} P(t)\right) \|x-y\| \\ &= \theta\|x-y\|, \quad n \geq T+1,\ x, y \in A(N, M), \end{aligned}$$

$$
\begin{aligned}
Sx(n) &= \frac{M(1+\bar{b})+N}{2} - b(n)x(n-\tau) \\
&\quad + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[f(t,x(t-c),x(t-d)) - a(t)] \\
&\le \frac{M(1+\bar{b})+N}{2} + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[|f(t,x(t-c),x(t-d))| + |a(t)|] \\
&\le \frac{M(1+\bar{b})+N}{2} + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[Q(t) + |a(t)|] \\
&\le \frac{M(1+\bar{b})+N}{2} + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}[Q(t) + |a(t)|] \\
&\le \frac{M(1+\bar{b})+N}{2} + \frac{M(1-\bar{b})-N}{2} \\
&= M, \quad n \ge T+1,\ x,y \in A(N,M)
\end{aligned}
$$

and

$$
\begin{aligned}
Sx(n) &= \frac{M(1+\bar{b})+N}{2} - b(n)x(n-\tau) + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty} f(t,x(t-c),x(t-d)) \\
&\ge \frac{M(1+\bar{b})+N}{2} - \bar{b}M - \sum_{s=T}^{n-1}\sum_{t=s}^{\infty} |f(t,x(t-c),x(t-d))| \\
&\ge \frac{M(1-\bar{b})+N}{2} - \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[Q(t) + |a(t)|] \\
&\ge \frac{M(1-\bar{b})+N}{2} - \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}[Q(t) + |a(t)|] \\
&\ge \frac{M(1-\bar{b})+N}{2} - \frac{M(1-\bar{b})-N}{2} \\
&= N, \quad n \ge T+1,\ x,y \in A(N,M),
\end{aligned}
$$

which imply (2.13). Thus the Banach fixed point theorem means that $S$ has a unique fixed point $x \in A(N,M)$, which is a bounded positive solution of Eq.(1.1).

By means of (2.13), (2.19), (2.20) and (2.22), we deduce that

$$
\begin{aligned}
&|x_{m+1}(n) - x(n)| \\
&= \Bigg|(1-\alpha_m)x_m(n) + \alpha_m\Bigg\{\frac{M(1+\bar{b})+N}{2} - b(n)x(n-\tau) \\
&\quad - \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[f(t,x_m(t-c),x_m(t-d)) - a(t)]\Bigg\} - x(n)\Bigg|
\end{aligned}
$$

$$\begin{aligned}
&\le (1-\alpha_m)|x_m(n)-x(n)| + \alpha_m|Sx_m(n)-Sx(n)| \\
&\le (1-\alpha_m)\|x_m-x\| + \alpha_m\theta\|x_m-x\| \\
&= (1-(1-\theta)\alpha_m)\|x_m-x\| \\
&\le e^{-(1-\theta)\alpha_m}\|x_m-x\|, \quad n \ge T+1,\, m \in \mathbb{N}_0,
\end{aligned}$$

which yields (2.14). It follows from (2.8) and (2.14) that $\lim_{m\to\infty} x_m = x$. This completes the proof. □

**Theorem 2.3.** *Assume that there exists a constant $\underline{b} \in (-1, 0]$ satisfying*

$$\underline{b} \le b(n) \le 0, \quad \textit{eventually.} \tag{2.23}$$

*Let $M$ and $N$ be two positive constants with $M(1+\underline{b}) > N$ and there exist two mappings $P, Q : \mathbb{N}_{n_0} \to \mathbb{R}^+$ satisfying* (2.2), (2.3), (2.16) *and* (2.17). *Then there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + |\beta|$ such that for any $x_0 \in A(N,M)$, the Mann iterative method $\{x_m\}_{m\ge 0}$ generated by*

$$x_{m+1}(n) = \begin{cases} (1-\alpha_m)x_m(n) + \alpha_m\big\{\frac{M(1+\underline{b})+N}{2} - b(n)x_m(n-\tau) \\ \quad + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[f(t, x_m(t-c), x_m(t-d)) - a(t)]\big\}, \\ \quad n \ge T+1,\, m \ge 0, \\ (1-\alpha_m)x_m(T+1) \\ \quad +\alpha_m\big\{\frac{M(1+\underline{b})+N}{2} - b(T+1)x_m(T+1-\tau) \\ \quad + \sum_{t=T}^{\infty}[f(t, x_m(t-c), x_m(t-d)) - a(t)]\big\}, \\ \quad \beta \le n \le T,\, m \ge 0 \end{cases} \tag{2.24}$$

*converges to a bounded positive solution $x \in A(N,M)$ of Eq.*(1.1) *and satisfies the error estimate* (2.7), *where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is a sequence in $[0,1]$ satisfying* (2.8).

*Proof.* It follows from (2.16), (2.17) and (2.23) that there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + |\beta|$ satisfying

$$\underline{b} \le b_n \le 0, \quad n \ge T; \tag{2.25}$$

$$\theta = |\underline{b}| + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty} P(t); \tag{2.26}$$

$$\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}[Q(t) + |a(t)|] < \frac{(1+\underline{b})M - N}{2}. \tag{2.27}$$

Define a mapping $S : A(N,M) \to l_\beta^\infty$ by

$$Sx(n) = \begin{cases} \frac{M(1+\underline{b})+N}{2} - b(n)x(n-\tau) \\ \quad + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[f(t, x(t-c), x(t-d)) - a(t)], \\ \quad n \ge T+1,\, x \in A(N,M), \\ Sx(T+1), \quad \beta \le n \le T,\, x \in A(N,M). \end{cases} \tag{2.28}$$

It follows from (2.3), (2.25), (2.27) and (2.28) that

$$
\begin{aligned}
&|Sx(n) - Sy(n)| \\
&= \Bigg| \frac{M(1+\bar{b})+N}{2} - b(n)x(n-\tau) + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[f(t,x(t-c),x(t-d)) - a(t)] \\
&\quad - \frac{M(1+\bar{b})+N}{2} + b(n)y(n-\tau) - \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[f(t,y(t-c),y(t-d)) - a(t)]\Bigg| \\
&\le b(n)|x(n-\tau) - y(n-\tau)| \\
&\quad + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}|f(t,x(t-c),x(t-d)) - f(t,y(t-c),y(t-d))| \\
&\le \bar{b}\|x-y\| + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}P(t)\max\{|x(t-c)-y(t-c)|,|x(t-d)-y(t-d)|\} \\
&\le \left(\bar{b} + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}P(t)\right)\|x-y\| \\
&= \theta\|x-y\|, \quad n \ge T+1,\ x,y \in A(N,M),
\end{aligned}
$$

$$
\begin{aligned}
Sx(n) &= \frac{M(1+\underline{b})+N}{2} - b(n)x(n-\tau) \\
&\quad + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[f(t,x(t-c),x(t-d)) - a(t)] \\
&\le \frac{M(1+\underline{b})+N}{2} - \underline{b}M + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[|f(t,x(t-c),x(t-d))| + |a(t)|] \\
&\le \frac{M(1-\underline{b})+N}{2} + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[Q(t) + |a(t)|] \\
&\le \frac{M(1-\underline{b})+N}{2} + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}[Q(t) + |a(t)|] \\
&\le \frac{M(1-\underline{b})+N}{2} + \frac{(1+\underline{b})M-N}{2} \\
&= M, \quad n \ge T+1,\ x \in A(N,M),
\end{aligned}
$$

and

$$
\begin{aligned}
Sx(n) &= \frac{M(1+\underline{b})+N}{2} - b(n)x(n-\tau) \\
&\quad + \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[f(t,x(t-c),x(t-d)) - a(t)]
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{M(1+\underline{b})+N}{2} - \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[|f(t,x(t-c),x(t-d))|+|a(t)|] \\
&\geq \frac{M(1+\underline{b})+N}{2} - \sum_{s=T}^{n-1}\sum_{t=s}^{\infty}[Q(t)+|a(t)|] \\
&\geq \frac{M(1+\underline{b})+N}{2} - \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}[Q(t)+|a(t)|] \\
&\geq \frac{M(1+\underline{b})+N}{2} - \frac{(1+\underline{b})M-N}{2} \\
&= N, \quad n \geq T+1,\ x \in A(N,M),
\end{aligned}
$$

which imply (2.13). It follows from (2.13) and the Banach fixed point theorem that $S$ has a unique fixed point $x \in A(N,M)$, which is a bounded positive solution of Eq.(1.1). The rest of the proof is similar to that of Theorem 2.2, and is omitted. This completes the proof. □

# References


[1] J.F. Cheng, Existence of a nonoscillatory solution of a second-order linear neutral difference equation, *Appl. Math. Lett.*, **20** (2007), 892–899.

http://dx.doi.org/10.1016/j.aml.2006.06.021

[2] W.T. Li and S.H. Saker, Oscillation of second-order sublinear neutral delay difference equations, *Appl. Math. Comput.*, **146** (2003), 543–551.

http://dx.doi.org/10.1016/S0096-3003(02)00604-5

[3] S.H. Saker, New oscillation criteria for second-order nonlinear delay difference equations, *Appl. Math. Comput.*, **142** (2003), 99–111.

http://dx.doi.org/10.1016/S0096-3003(02)00286-2

[4] B.G. Zhang, Oscillation and asymptotic behavior of second order difference equations, *J. Math. Anal. Appl.*, **173** (1993), 58–68.

http://dx.doi.org/10.1006/jmaa.1993.1052


**Received: January 12, 2015; Published March 4, 2015**