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Generalized Numerical Range

S. Bouali and M. Ech-chad

Department of Mathematics
Faculty of Sciences Kénitra, B.P. 133 Kénitra, Morocco

Abstract

In this paper we will introduce the generalized numerical range $W_g(A)$ of an operator A on a separable Hilbert space. We will give some properties of $W_g(A)$, and study the situation in which $W_g(A) = W(A)$ (the ordinary numerical range of A). We also shed light on the generalized numerical range of derivation.

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Introduction

Let $\mathcal{A}$ be a complex Banach algebra with identity e, and let $P = \{f \in \mathcal{A}^*,\ f(e) = 1 = \|f\|\}$ be the set of states on $\mathcal{A}$. The numerical range [7] of an element A in $\mathcal{A}$ is by definition the set;

$$W_o(A) = \{f(A),\ f \in P\}.$$

$W_o(A)$ is convex, compact and contains the spectrum of A [7].
If $\mathcal{A} = \mathcal{L}(H)$ is the algebra of bounded operators on a Hilbert space H, then $W_o(A) = \overline{W(A)}$ is precisely the closure of the ordinary numerical range,

$$W(A) = \{< Ax, x >,\ \|x\| = 1\}.$$

The numerical range was systematically studied by several authors, for example F. Bonsall and J. Ducan [2], K. Gustafson and D. Rao [4], and P. Halmos [5]. For $A \in \mathcal{A}$, we define the generalized numerical range of A by

$$W_{og}(A) = \{f(A),\ f \in P_g\},$$

where $P_g = \{f \in \mathcal{A}^*;\ f(e) = \|f\| \leq 1\}$. If $A \in \mathcal{L}(H)$, let

$$W_g(A) = \{< Ax, x >;\ \|x\| \leq 1\}.$$

The motivation of the numerical range which has just been introduced is based on one of Halmos's results [5].
In the first part, we give some properties of the generalized numerical range. We prove that $W_g(A)$ is convex, and obtain some equivalent (sufficient) conditions for $W_g(A) = W(A)$.
It has been shown in theorem 2 [1] that if $A \in \mathcal{L}(H)$ is a compact normal operator, then $W(A) = co(\sigma_p(A))$, the convex hull of the point spectrum of A. In the second part, we show that for a compact operator A; $W_g(A)$ is closed, and obtain that for a compact normal operator A; $W_g(A) = co(\sigma_p(A) \cup \{0\})$. In the third part, we prove that, if for all λ in $\mathbb{C}$, $\|A-\lambda\| = \rho(A-\lambda)$ (ρ stands for the spectral radius) and $\|B-\lambda\| = \rho(B-\lambda)$, then the generalized numerical range $W_{og}(\delta_{AB}) = co(\sigma(\delta_{AB}) \cup \{0\})$. δ_{AB} is the generalized derivation operator associated with $(A, B) \in (\mathcal{L}(H))^2$, defined on $\mathcal{L}(H)$ by $\delta_{AB}(X) = AX - XB$ for all $X \in \mathcal{L}(H)$.
In addition to the notation already introduced, we shall use the following notation. We shall denote the ideal of all compact operators by $\mathcal{K}(H)$. Given $X \in \mathcal{L}(H)$, the spectrum, the point spectrum and the spectral radius of X will be denoted by $\sigma(X)$, $\sigma_p(X)$ and $\rho(X)$ respectively.

1 Properties of Generalized Numerical Range

Theorem 1.1 *If $A \in \mathcal{L}(H)$; then $W_{og}(A) = \overline{W_g(A)}$.*

Proof. $\lambda \in W_{og}(A)$ is equivalent to; there exists $f \in P_g$ such that $\frac{\lambda}{\|f\|} \in W_o(A) = \overline{W(A)}$. This occurs if and only if $\lambda \in \overline{W_g(A)}$. ♢

Theorem 1.2 *If $A \in \mathcal{L}(H)$, then $W_g(A)$ is convex.*

Proof. Let $< Ax, x >$ and $< Ay, y >$ two elements in $W_g(A)$, $x_1 = \frac{x}{\|x\|}$, $y_1 = \frac{y}{\|y\|}$ and $\alpha = \sqrt{t\|x\|^2 + (1-t)\|y\|^2}$ where $t \in [0, 1]$. Then $< Ax_1, x_1 >$ and $< Ay_1, y_1 > \in W(A)$. Since $W(A)$ is a convex set [5, solution 166, p. 317], there exists $u \in H$; such that $\|u\| = 1$ and

$$t(\frac{\|x\|}{\alpha})^2 < Ax_1, x_1 > +(1-t)(\frac{\|y\|}{\alpha})^2 < Ay_1, y_1 > = < Au, u > .$$

Hence

$$t < Ax, x > +(1-t) < Ay, y >=< A(\alpha u), \alpha u >,$$

and $\|\alpha u\| = |\alpha| \leq 1$. Thus $W_g(A)$ is convex. $\diamondsuit$

Corollary 1.1 *For $A \in \mathcal{L}(H)$, $W_{og}(A)$ is convex, compact and contains the spectrum of A.*

Remarks 1.1
(1) *Remark that For all $A \in \mathcal{L}(H)$; $0 \in W_g(A)$.*
(2) *It is clear that $W_g(I) = [0,1]$ and $W(I) = \{1\}$. Thus $W(A)$ is properly contained in $W_g(A)$ in general.*
(3) *If $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, then a simple calculation shows that $W_g(T) = [0,2]$ and $W(T) = [1,2]$. Thus $W(T)$ is a segment which does not contain 0.*

Theorem 1.3 *If $A \in \mathcal{L}(H)$, then the following assertions are equivalent:*
(1) $W_g(A) = W(A)$;
(2) $0 \in W(A)$.

Proof. Since $0 \in W_g(A)$, it is sufficient to show that (2) $\Rightarrow$ (1). Assume that $0 \in W(A)$. Let $\lambda =< Ay, y >\in W_g(A)$. Note that $y = tx$ with $|t| \leq 1$ and $\|x\| = 1$. Since $W(A)$ is convex [5, solution 166, p. 317] and

$$\lambda = t^2 < Ax, x > +(1-t^2)0,$$

it follows that $\lambda \in W(A)$. $\diamondsuit$

Corollary 1.2 *Let $A \in \mathcal{L}(H)$. If there exists $\lambda \in \sigma_p(A)$ and $r \in \mathbb{R}^-$, such that $r\lambda \in \sigma_p(A)$, then $W_g(A) = W(A)$.*

Proof. Suppose that there exists $\lambda \in \sigma_p(A)$ and $r \in \mathbb{R}^-$, such that $r\lambda \in \sigma_p(A)$. Let $t = \frac{-r}{1-r} \in [0,1]$. A simple calculation shows that:

$$0 = t\lambda + (1-t)r\lambda.$$

Since $W(A)$ is convex [5, solution 166, p. 317], $0 \in W(A)$. Thus $W_g(A) = W(A)$. $\diamondsuit$

Proposition 1.1 *Let $A \in \mathcal{L}(H)$ and $\lambda \in \mathbb{C}$, such that $|\lambda| = \|A\|$. If $\lambda \in W_g(A)$, then the point spectrum of A is not empty.*

Proof. Suppose that $\lambda =< Ay, y >$ where $\|y\| \leq 1$. Then we have:

$$\|A\| = |\lambda| \leq | < Ay, y > | \leq \|Ay\|\|y\| \leq \|A\|.$$

It follows that $| < Ay, y > | = \|Ay\|\|y\|$. Hence there exists $\mu \in \mathbb{C}$ such that $Ay = \mu y$. Consequently $\lambda =< Ay, y >= \mu\|y\|^2$, which implies that $y \neq 0$. Thus $\mu \in \sigma_p(A)$. $\diamondsuit$

2 Generalized Numerical Range of Compact Operators

Theorem 2.1 *If $A \in \mathcal{K}(H)$, then $W_g(A)$ is closed.*

Proof. Let $\lambda \in \overline{W_g(A)}$, then there exists a sequence $(< Ax_n, x_n >)_n$, where $\|x_n\| \leq 1$ for all n, converging to λ. Since the unit ball is weakly compact, there exists a subsequence $(x_{n_k})_k$ which is weakly convergent to an x where $\|x\| \leq 1$. Since A is a compact operator, $(A(x_{n_k}))_k$ is strongly convergent to Ax.
However,

$$| < Ax_{n_k}, x_{n_k} > - < Ax, x > | \leq \|x_{n_k}\| \|Ax_{n_k} - Ax\| + | < x_{n_k}, Ax > - < x, Ax > |.$$

Therefore $(< Ax_{n_k}, x_{n_k} >)_k$ converge to $< Ax, x >$. Thus $\lambda =< Ax, x >\in W_g(A)$. ♢

Remark 2.1
(1) *Let $(e_n)_{n\geq 0}$ be an orthonormal basis for H, and S the unilateral shift defined by $Se_n = e_{n+1}$. It is known that $W(S) = D = \{z \in \mathbb{C} \ / \ |z| < 1\}$. Since $0 \in W(S)$, it follows from theorem 1.3 that $W_g(S) = W(S) = D$. This shows that $W_g(A)$ is not closed in general.*
(2) *$W_g(I) = [0, 1]$ is closed but I is not compact. Thus the condition $A \in \mathcal{K}(H)$ is not necessary.*

Theorem 2.2 *For a compact normal operator A on H, $W_g(A) = co(\sigma_p(A) \cup \{0\})$.*

Proof. Clearly $co(\sigma_p(A) \cup \{0\}) \subset W_g(A)$, so it is sufficient to show that $W_g(A) \subset co(\sigma_p(A) \cup \{0\})$. Let $\lambda =< Ax, x >\neq 0$ where $\|x\| \leq 1$. If $y = \frac{x}{\|x\|}$, then

$$< Ay, y >\in W(A) = co(\sigma_p(A)) \ [1, \ theorem \ 2].$$

Thus

$$\lambda = \|x\|^2 < Ay, y > +(1 - \|x\|^2)0 \in co(\sigma_p(A) \cup \{0\}).$$

Therefore, $W_g(A) \subset co(\sigma_p(A)\{0\})$. ♢

Remark 2.2
It is clear that $W_g(I) = [0, 1] = co(\sigma_p(I) \cup \{0\})$, but I is not compact. Thus the condition $A \in \mathcal{K}(H)$ is not necessary.

3 Generalized Numerical Range of Derivation

Theorem 3.1 *If $A \in \mathcal{L}(H)$ such that $\|A-\lambda\| = \rho(A-\lambda)$, for all λ in $\mathbb{C}$, then $W_{og}(A) = co(\sigma(A) \cup \{0\})$.*

Proof. Let $\lambda = f(A)$, where $f \in P_g$. If $f = \|f\|g$, we can write $g(A) \in W_o(A) = co(\sigma(A))$ [3, p. 564]. Thus

$$\lambda = \|f\|g(A) + (1 - \|f\|)0 \in co(\sigma(A) \cup \{0\}).$$

Therefore, $W_{og}(A) \subset co(\sigma(A) \cup \{0\})$. Using corollary 1.1, we obtain $W_{og}(A) = co(\sigma(A) \cup \{0\})$. $\diamondsuit$

Corollary 3.1 *For a hyponormal operator A on H, $W_{og}(A) = co(\sigma(A) \cup \{0\})$.*

Proof. Using theorem 1 [6] we have $\|A-\lambda\| = \rho(A-\lambda)$, for all λ in $\mathbb{C}$. It follows from theorem 3.1 that $W_{og}(A) = co(\sigma(A) \cup \{0\})$. $\diamondsuit$

Theorem 3.2 *If A, $B \in \mathcal{L}(H)$ such that $\|A-\lambda\| = \rho(A-\lambda)$ and $\|B-\lambda\| = \rho(B-\lambda)$ for all λ in $\mathbb{C}$, then $W_{og}(\delta_{AB}) = co(\sigma(\delta_{AB}) \cup \{0\})$.*

Proof. Let $\lambda = f(\delta_{AB})$, where $f \in P_g$. If we write $f = tg$ with $t \in [0,1]$ and $\|g\| = 1$, then $g(\delta_{AB}) \in W_o(\delta_{AB}) = co(\sigma(\delta_{AB}))$ [3, p. 565]. Thus

$$\lambda = tg(\delta_{AB}) + (1-t)0 \in co(\sigma(\delta_{AB}) \cup \{0\}).$$

Therefore, $W_{og}(\delta_{AB}) \subset co(\sigma(\delta_{AB}) \cup \{0\})$. Since $W_{og}(\delta_{AB})$ is convex and contains the spectrum of δ_{AB}, it follows that $W_{og}(\delta_{AB}) = co(\sigma(\delta_{AB}) \cup \{0\})$. $\diamondsuit$

Corollary 3.2 *For hyponormal operators A and B on H, $W_{og}(\delta_{AB}) = co(\sigma(\delta_{AB}) \cup \{0\})$.*

Proof. Using theorem 1 [6] we have $\|A-\lambda\| = \rho(A-\lambda)$ and $\|B-\lambda\| = \rho(B-\lambda)$, for all λ in $\mathbb{C}$. It follows from theorem 3.2 that $W_{og}(\delta_{AB}) = co(\sigma(\delta_{AB}) \cup \{0\})$. $\diamondsuit$

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