Weighted Composition Followed by Differentiation between Weighted Bergman Space and $H^\infty$ on the Unit Ball

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Abstract

We define differentiation operator on $H(\mathbb{B})$ by radial derivative, then we study the boundedness and compactness of products of multiplication operator, composition operator and differentiation operator acting between weighted Bergman spaces and $H^\infty$ on the unit ball.

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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane. Let $\mathbb{B} = \{ z \in \mathbb{C}^n : |z| < 1 \}$ be the unit ball of $\mathbb{C}^n$, and $S = \partial \mathbb{B}$ its boundary. We will denote by $dv$ the normalized Lebesgue measure on $\mathbb{B}$.

Recall that for $\alpha > -1$ the weighted Lebesgue measure $dv_\alpha$ is defined by

$$dv_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha dv(z),$$

where

$$c_\alpha = \frac{\Gamma(n + 1 + \alpha)}{n! \Gamma(1 + \alpha)}$$

is a normalizing constant so that $dv_\alpha$ is a probability measure on $\mathbb{B}$.

Let $H(\mathbb{B})$ denote the space of holomorphic functions on $\mathbb{B}$. Take $1 \leq p < \infty$. Then $f \in H(\mathbb{B})$ is said to be in the weighted Bergman space $A^p_\alpha(\mathbb{B})$ if

$$\|f\|_{A^p_\alpha} = \int_{\mathbb{B}} |f(z)|^p dv_\alpha(z) < \infty.$$

As we all know,

$$H^\infty = \{ f \in H(\mathbb{B}) : \|f\|_\infty = \sup_{z \in \mathbb{B}} |f(z)| < \infty \}.$$

Let $\varphi$ be an analytic self-mapping of $\mathbb{B}$, then the composition operator on $H(\mathbb{B})$ is given by

$$C_\varphi f = f \circ \varphi.$$

Recently, there have been an increasing interest in studying composition operators acting on different spaces of analytic functions, for example, see [3,5] for details about composition operators on classical spaces of analytic functions.

Let $D$ be the differentiation operator defined by

$$Df = f', \quad f \in H(\mathbb{D}).$$

Hibschweiler and Portnoy [7] defined the linear operators $DC_\varphi$ and $C_\varphi D$ and investigated the boundedness and compactness of these operators between Bergman spaces using Carleson-type measure. S. Ohno [10] discussed boundedness and compactness of $C_\varphi D$ between Hardy spaces. Recall the multiplication operator $M_\psi$ defined by

$$M_\psi f = \psi f, \quad f \in H(\mathbb{D}).$$

A. K. Sharma defined [1] products of these operators in the following six ways:

$$(M_\psi C_\varphi Df)(z) = \psi(z)f'(\varphi(z)),$$

$$(M_\psi DC_\varphi f)(z) = \psi(z)(\varphi'(z))f'(\varphi(z)),$$

$$(C_\varphi M_\psi Df)(z) = \psi(\varphi(z))f'(\varphi(z)).$$
Then we also have six ways of products of these operators on the unit ball:

\[
(DM_\varphi C_\varphi f)(z) = \varphi'(z) f(\varphi(z)) + \psi(z) (\varphi'(z)) f'(\varphi(z)),
(C_\varphi DM_\varphi f)(z) = \varphi'(\varphi(z)) f(\varphi(z)) + \psi(\varphi(z)) f'(\varphi(z)),
(DC_\varphi M_\varphi f)(z) = \varphi'(\varphi(z)) f(\varphi(z)) \varphi'(z) + \psi(\varphi(z)) f'(\varphi(z)) \varphi'(z).
\]

for \(z \in \mathbb{D}\) and \(f \in H(\mathbb{D})\).

There are a lot of papers researching these products, see [2,4,9]. Since those results focus on \(\mathbb{D}\), naturally, we consider similar questions on \(\mathbb{B}\). Of course, the method we used is different from the case on \(\mathbb{D}\).

For \(f \in H(\mathbb{B})\), we define the differentiation operator on \(H(\mathbb{B})\) by radial derivative. Recall that for \(z \in \mathbb{B}\) and \(f \in H(\mathbb{B})\),

\[
Rf = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z) = \lim_{r \to 0} \frac{f(z + rz) - f(z)}{r}, \quad r \in \mathbb{R}.
\]

One can see that for \(z \neq \varphi^{-1}(0)\),

\[
R(f \circ \varphi)(z) = \frac{(Rf)(\varphi(z)) \cdot R\varphi(z)}{\varphi(z)}.
\]

Then we also have six ways of products of these operators on the unit ball:

\[
(M_\psi C_\varphi R)f(z) = \psi(z) \cdot (Rf)(\varphi(z)),
(C_\varphi M_\psi Rf)(z) = \varphi(\varphi(z)) \cdot (Rf)(\varphi(z)),
(M_\psi RC_\varphi f)(z) = \frac{\psi(z) \cdot R\varphi(z) \cdot (Rf)(\varphi(z))}{\varphi(z)},
(C_\varphi RM_\psi f)(z) = (R\psi)(\varphi(z)) \cdot f(\varphi(z)) + \psi(\varphi(z)) \cdot (Rf)(\varphi(z)),
(RM_\psi C_\varphi f)(z) = f(\varphi(z)) \cdot R\psi(z) + \frac{\psi(z) \cdot R\varphi(z) \cdot (Rf)(\varphi(z))}{\varphi(z)},
(RC_\varphi M_\psi f)(z) = \frac{(R\psi)(\varphi(z)) \cdot R\varphi(z) \cdot f(\varphi(z)) + \psi(\varphi(z)) \cdot (Rf)(\varphi(z)) \cdot R\varphi(z)}{\varphi(z)}
\]

for \(z \neq \varphi^{-1}(0)\).

In this paper, we characterize the boundedness and compactness of \(RM_\psi C_\varphi\) between weighted Bergman spaces and \(H^\infty\) on the unit ball, which extend the results of H. Li in [4].

2. Main results

The following lemma is the Theorem 20 in [8].

**Lemma 2.1.** Suppose \(p > 0\), \(n + 1 + \alpha > 0\), then there exist a constant \(C > 0\) (depend on \(p\) and \(\alpha\)) such that

\[
|f(z)| \leq \frac{C\|f\|_{A_p^\alpha}}{(1 - |z|^2)^{\frac{n+1+\alpha}{p}}}
\]
for all \( f \in A^p_\alpha \), \( z \in \mathbb{B} \).

Recall that for a holomorphic function \( f \) in \( \mathbb{B} \) we write
\[
\nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z) \right)
\]
and call \( \nabla f(z) \) the gradient of \( f \) at \( z \).

Then we give the following lemma.

**Lemma 2.2.** Let \( p > 0 \), \( \alpha > -1 \), then there exist a constant \( C > 0 \) such that
\[
|Rf(z)| \leq C|z|\|f\|_{A^2_\alpha(\mathbb{B})} \frac{1}{(1 - |z|^2)^{\frac{n+1+\alpha}{2} + 1}}
\]
for all \( f \in A^2_\alpha(\mathbb{B}) \) and \( z \in \mathbb{B} \).

**Proof.** For a \( f \) in \( A^2_\alpha(\mathbb{B}) \), from the Exercise 3.5.9 in [3] we know that
\[
\|f\|_{A^2_\alpha(\mathbb{B})} \cong |f(0)|^2 + \int_\mathbb{B} |\nabla f(z)|^2(1 - |z|^2)^{\alpha + 2} dv(z).
\]
On the other hand,
\[
\|f\|_{A^2_\alpha(\mathbb{B})}^2 = \int_\mathbb{B} |f(z)|^2(1 - |z|^2)^\alpha dv(z).
\]
Then by Lemma 2.1,
\[
|Rf(z)| \leq \frac{C\|f\|_{A^2_\alpha(\mathbb{B})}}{(1 - |z|^2)^{\frac{n+1+\alpha}{2} + 1}}.
\]
By the proof of Lemma 2.14 in [6] we have
\[
|Rf(z)| \leq |z|\|\nabla f(z)\|,
\]
then we draw the conclusion. \(\square\)

**Lemma 2.3.** Suppose \( p > 0 \), \( \alpha > -1 \), \( \varphi : \mathbb{B} \to \mathbb{B} \) be analytic, \( \psi \in H(\mathbb{B}) \),
set \( T = RM_\psi C_\varphi : A^2_\alpha(\mathbb{B}) \to H^\infty \). Then \( T \) is compact if and only if \( T \) is bounded and for any bounded sequence \( \{f_k\} \) in \( A^2_\alpha(\mathbb{B}) \) which converges to zero uniformly on compact subsets of \( \mathbb{B} \), \( \|Tf_k\|_\infty \to 0 \) as \( k \to \infty \).

By standard arguments from Proposition 3.11 in [3], this lemma follows.

**Theorem 2.4.** Assume that \( p > 0 \), \( \alpha > -1 \), \( \varphi : \mathbb{B} \to \mathbb{B} \) be analytic , \( \psi \in H(\mathbb{B}) \). Then \( RM_\psi C_\varphi : A^2_\alpha(\mathbb{B}) \to H^\infty \) is bounded if and only if
\[
\sup_{z \in \mathbb{B}} \frac{|\psi(z)||R\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1+\alpha}{2} + 1}} < \infty, \tag{1}
\]
\[
\sup_{z \in \mathbb{B}} \frac{|R\psi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1+\alpha}{2}}} < \infty. \tag{2}
\]
Proof. Assume that $RM\varphi C\varphi : A_\alpha^2(\mathbb{B}) \to H^\infty$ is bounded. For every $a \in \mathbb{B}$, let

$$f_a(z) = \left(\frac{a - z}{1 - \overline{a}z}\right)^2 \left(\frac{1 - |a|^2}{(1 - \overline{a}z)^2}\right) \frac{n + 1 + \alpha}{2}.$$ 

Then a change of variables yields that

$$\|f_a\|_{A_\alpha^2(\mathbb{B})} = \int \|f_a(z)\|^2 (1 - |z|^2)^\alpha dv(z)$$

$$= \int_\mathbb{B} \frac{|a - z|^2 (1 - |a|^2)^\alpha (1 - |z|^2)^\alpha}{1 - \overline{a}z^{2\alpha}} \left(\frac{1 - |a|^2}{(1 - \overline{a}z)^2}\right)^{n+1} dv(z)$$

$$\leq \int_\mathbb{B} 2^{2\alpha} \frac{1 - |a|^2}{(1 - \overline{a}z)^2}^{n+1} dv(z) \leq C.$$ 

Obviously, $f_a \in A_\alpha^2(\mathbb{B})$, $\sup_{a \in \mathbb{B}} \|f_a\|_{A_\alpha^2(\mathbb{B})} < C$. It is easy to see that

$$RF_a(z) = \left(\frac{a - z}{1 - \overline{a}z}\right) \frac{(n + 1 + \alpha)(1 - |a|^2)^\frac{n + 1 + \alpha}{2}}{(1 - \overline{a}z)^{n+2+\alpha}}$$

$$+ \left(\frac{1 - |a|^2}{(1 - \overline{a}z)^2}\right)^\frac{n + 1 + \alpha}{2} \frac{(a - z) - z(1 - \overline{a}z)}{(1 - \overline{a}z)^2},$$

which implies that

$$f_{\varphi(z)}(\varphi(z)) = 0,$$

$$(RF_{\varphi(z)})(\varphi(z)) = \frac{-\varphi(z)}{(1 - |\varphi(z)|^2)^\frac{n + 1 + \alpha}{2} + 1}.$$ 

Then we have

$$\|RM\varphi C\varphi f_{\varphi(z)}\|_\infty \geq \sup_{z \in \mathbb{B}} |f_{\varphi(z)}(\varphi(z)) \cdot R\psi(z) + \frac{\psi(z) \cdot R\varphi(z) \cdot (RF_{\varphi(z)})(\varphi(z))}{\varphi(z)}|$$

$$\geq \sup_{z \in \mathbb{B}} |\psi(z)| |R\varphi(z)| (1 - |\varphi(z)|^{n + 1 + \alpha}).$$

Then (1) can be obtained.

Next we assume

$$g_a(z) = \left(\frac{1 - |a|^2}{(1 - \overline{a}z)^2}\right)^\frac{n + 1 + \alpha}{2}.$$ 

Similarly as above, $g_a \in A_\alpha^2(\mathbb{B})$, $\sup_{a \in \mathbb{B}} \|g_a\|_{A_\alpha^2(\mathbb{B})} < C$. On the other hand,

$$g_{\varphi(z)}(\varphi(z)) = \frac{1}{(1 - |\varphi(z)|^2)^\frac{n + 1 + \alpha}{2}},$$

$$(Rg_{\varphi(z)})(\varphi(z)) = \frac{(n + 1 + \alpha)|\varphi(z)|^2}{(1 - |\varphi(z)|^2)^\frac{n + 1 + \alpha}{2} + 1},$$

which implies that

$$\|RM\varphi C\varphi g_{\varphi(z)}\|_\infty \geq \sup_{z \in \mathbb{B}} |g_{\varphi(z)}(\varphi(z)) \cdot R\psi(z) + \frac{\psi(z) \cdot R\varphi(z) \cdot (Rg_{\varphi(z)})(\varphi(z))}{\varphi(z)}|.$$
For all

From Theorem 2.4 we know \( \psi \)

Theorem 2.5. Assume that \( \phi \)

Since \( H \)

Proof. From Lemma 2.1 and Lemma 2.2,

\[
\| (R\psi f)(z) \| = \sup_{z \in B} \left| f(\phi(z)) \cdot R\psi(z) + \frac{\psi(z) \cdot R\psi(z) \cdot (Rf)(\phi(z))}{\phi(z)} \right|
\]

\[
\leq \sup_{z \in B} \frac{C |R\psi(z)||f|_{A_2^a(B)}}{(1 - |\phi(z)|^2)^{n+\frac{1+a}{2}}} + \sup_{z \in B} \frac{C |\phi(z)||R\psi(z)||f|_{A_2^a(B)}}{(1 - |\phi(z)|^2)^{n+\frac{1+a}{2}+1}}.
\]

From the assumption, \( R\psi f : A_2^a(B) \to H^\infty \) is bounded.

\[\square\]

**Theorem 2.5.** Assume that \( p > 0 \), \( \alpha > -1 \), \( \phi : B \to B \) be analytic , \( \psi \in H(B) \). Then \( R\psi f : A_2^a(B) \to H^\infty \) is compact if and only if

\[
\sup_{z \in B} |\psi(z)||R\psi(z)| < \infty,
\]

(3)

\[
\sup_{z \in B} |R\psi(z)| < \infty,
\]

(4)

\[
\sup_{|z(z)| \to 1} \frac{|\psi(z)||R\psi(z)|}{(1 - |\phi(z)|^2)^{n+\frac{1+a}{2}+1}} = 0,
\]

(5)

\[
\sup_{|z(z)| \to 1} \frac{|R\psi(z)|}{(1 - |\phi(z)|^2)^{n+\frac{1+a}{2}}} = 0.
\]

(6)

**Proof.** Assume that \( R\psi f : A_2^a(B) \to H^\infty \) is compact. From Lemma 2.3, we know that \( R\psi f \) is bounded and by taking \( f(z) = 1 \), it follows that \( \sup_{z \in B} |R\psi(z)| < \infty \). By taking the function \( f(z) = z \), we have that

\[
\sup_{z \in B} \left| \psi(z) \cdot R\psi(z) + R\psi(z) \cdot \phi(z) \right| < \infty.
\]

From above and \( \| \phi \|_{\infty} < 1 \), we get \( \sup_{z \in B} |\psi(z) \cdot R\psi(z)| < \infty \).

If we suppose (5) does not hold, then there exist a positive number \( \delta \in (0, 1) \) and a sequence \( \{z_n\}_{n \in \mathbb{N}} \) in \( B \), and \( |\phi(z_n)| \to 1 \) as \( n \to \infty \) such that

\[
\sup_{|z(z)| \to 1} \frac{|\psi(z_n)||R\psi(z_n)|}{(1 - |\phi(z_n)|^2)^{n+\frac{1+a}{2}+1}} \geq \delta
\]

for all \( n \in \mathbb{N} \). Next consider function

\[
f_n(z) = \frac{\phi(z_n) - z}{1 - \phi(z_n)z} \left( \frac{1 - |\phi(z_n)|^2}{(1 - \phi(z_n)z)^2} \right)^{\frac{n+1+a}{2}}, \quad z \in B.
\]

From Theorem 2.4 we know \( f_n \in A_2^a(B) \), \( f_n \to 0 \) uniformly on compact subsets of \( B \). From Lemma 3, it follows that a sequence \( \{R\psi f_n\} \) tends to 0 in \( H^\infty \).
On the other hand,

\[\| (RM_\psi C_\varphi f_n)(z_n) \|_\infty \geq |f_n(\varphi(z_n)) \cdot R\psi(z_n) + \frac{\psi(z_n) \cdot R\varphi(z_n) \cdot (RF_n)(\varphi(z_n))}{\varphi(z_n)}| \]

\[\geq \frac{C|\psi(z_n)||R\varphi(z_n)|}{(1 - |\varphi(z_n)|^2)^{\frac{n+1+\alpha}{2}}} \geq \delta.\]

Which is absurd, so (5) holds.

Now we consider the other function

\[g_k(z) = \left(1 - |\varphi(z_k)|^2\right)^{\frac{n+1+\alpha}{2}}, \quad z \in \mathbb{B}.\]

As the proof of Theorem 2.4, \(g_k \in \mathbb{A}_\alpha^2(\mathbb{B})\), \(g_k \to 0\) uniformly on compact subsets of \(\mathbb{B}\). From Lemma 2.3, we can see that the sequence \(RM_\psi C_\varphi g_k \to 0\) in \(H^\infty\).

\[\| (RM_\psi C_\varphi g_k)(z_k) \|_\infty \geq |g_k(\varphi(z_k)) \cdot R\psi(z_k) + \frac{\psi(z_k) \cdot R\varphi(z_k) \cdot (Rg_k)(\varphi(z_k))}{\varphi(z_k)}| \]

\[\geq \frac{C|\psi(z_k)||R\varphi(z_k)|}{(1 - |\varphi(z_k)|^2)^{\frac{n+1+\alpha}{2}}} - \frac{|R\psi(z_k)|}{(1 - |\varphi(z_k)|^2)^{\frac{n+1+\alpha}{2}}}.\]

Thus

\[\lim_{|\varphi(z_k)| \to 1} \frac{|\psi(z_k)||R\varphi(z_k)|}{(1 - |\varphi(z_k)|^2)^{\frac{n+1+\alpha}{2}}} = \lim_{|\varphi(z_k)| \to 1} \frac{|R\psi(z_k)|}{(1 - |\varphi(z_k)|^2)^{\frac{n+1+\alpha}{2}}}.\]

From the above proof, we get (6).

Conversely, for any bounded sequence \(\{f_n\} \in \mathbb{A}_\alpha^2(\mathbb{B})\) with \(\{f_n\} \to 0\) uniformly on compact subsets of \(\mathbb{B}\), by Lemma 2.3, it is enough to prove \(RM_\psi C_\varphi g_k\) tends to 0 in \(H^\infty\). From (5) and (6), given every \(\varepsilon > 0\), there exists a \(\delta \in (0, 1)\) such that when \(\delta < |\varphi(z)| < 1,\)

\[\frac{|\psi(z)||R\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1+\alpha}{2}}} < \varepsilon; \quad \frac{|R\psi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1+\alpha}{2}}} < \varepsilon.\]

On the other hand, since \(\{f_n\} \to 0\) uniformly on compact subsets of \(\mathbb{B}\), if we set \(h_n(z) = R\frac{f_n(z)}{z}\), then Cauchy’s estimates give that \(\{h_n\} \to 0\) uniformly on compact subsets of \(\mathbb{B}\), thus there exists an \(N_0 \in \mathbb{N}\), for every \(n > N_0, \ |\varphi(z)| \leq \delta, \ |f_n(\varphi(z))| < \varepsilon\) and \(|h_n(\varphi(z))| < \varepsilon.\)

Thus, from (3) and (4) we have

\[\sup_{|\varphi(z)| \leq \delta} |f_n(\varphi(z)) \cdot R\psi(z) + \frac{\psi(z) \cdot R\varphi(z) \cdot (RF_n)(\varphi(z))}{\varphi(z)}| \]

\[\leq \varepsilon \sup_{|\varphi(z)| \leq \delta} |\psi(z) \cdot R\varphi(z)| + \sup_{|\varphi(z)| \leq \delta} |R\psi(z)| \leq C\varepsilon.\]
Thus
\[
\|RM_\psi C_\varphi f_n\|_\infty = \sup_{z \in \mathbb{B}} \left| f_n(\varphi(z)) \cdot R\psi(z) + \frac{\psi(z) \cdot R\varphi(z) \cdot (Rf_n)(\varphi(z))}{\varphi(z)} \right|
\]
\[
\leq \sup_{\delta < |\varphi(z)| < 1} \left| f_n(\varphi(z)) \cdot R\psi(z) + \frac{\psi(z) \cdot R\varphi(z) \cdot (Rf_n)(\varphi(z))}{\varphi(z)} \right|
\]
\[
+ \sup_{|\varphi(z)| \leq \delta} \left| f_n(\varphi(z)) \cdot R\psi(z) + \frac{\psi(z) \cdot R\varphi(z) \cdot (Rf_n)(\varphi(z))}{\varphi(z)} \right|
\]
\[
\leq C \varepsilon + 2 \varepsilon.
\]
So \(RM_\psi C_\varphi : A^2_\alpha(\mathbb{B}) \to H^\infty\) is compact. \(\square\)

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