Common Local Spectral Properties of Bounded Linear Operators $AT$ and $SA$

such that $BSA = ATB$

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Abstract

Let $L(X,Y)$ the Banach space of linear continuous operators between Banach spaces $X$ and $Y$. The main goal of this paper is to study some common local spectral properties of the linear operators $AT$ and $SA$ such that $T,S \in L(Y,X)$ and $A,B \in L(X,Y)$ satisfying the operator equation $BSA = ATB$. We prove some local spectral inclusions. Again some results concerning the commutator $C(S,T)$ at $A \in L(X,Y)$ are given.

Keywords: local spectral theory, SVEP Property, Bishop’s Property ($\beta$), Operator equation, Commutator
1 Introduction

Let $X$ and $Y$ the Banach spaces, $L(X,Y)$ denotes the set of linear continuous operators from $X$ to $Y$. If $X = Y$ write $L(X) = L(X,X)$ the Banach algebra of all bounded linear operators on $X$. The dual space of $X$ is denoted by $X^*$. For $T \in L(X)$ we denote by $\sigma(T)$, $\mathcal{R}(T)$, $\mathcal{N}(T)$ and $T^*$ respectively the spectrum, the range, the kernel and the adjoint of $T$.

Recall that several authors has investigated to study the common spectral properties of two given operators $T \in L(X)$ and $S \in L(Y)$ and linked in some way by an operator $A \in L(X,Y)$ for example $SA = AT$. It is known that $TA$ and $SA$ share many common local spectral properties such as SVEP property, property of Bishop ($\beta$), property ($\delta$), Decomposability property, property ($\beta_\epsilon$), Property of Dunford ($C$). See [2], [3], [7], [10], and [11].

Recently [12] [13], Q.P. Zeng and H.J. Zhong showed that under the condition $ASA = ATA$ the operators $AT$ and $SA$ share many common various regularities in the sense of Kordula and Müller [9].

In this paper, we propose to study the relation between local spectral properties of the operators $AT$ and $SA$ under the general asymptotic equation $BSA = ATB$, noting that $B$ may be different from $A$.

Firstly we fix some concept in local spectral theory. An operator $T \in L(X)$ has the single valued extension property (SVEP, for short) at $\lambda \in \mathbb{C}$ if and only if for every neighborhood $U_{\lambda}$ of $\lambda$ the only analytic function $f : U_{\lambda} \to X$ satisfying $(T - \mu)f(\mu) = 0 \ \forall \mu \in U_{\lambda}$ is the null function $f \equiv 0$. We say that $T$ has SVEP if $T$ has SVEP for all $\lambda \in \mathbb{C}$, see [8].

For $T \in L(X)$, the set of local resolvent $\rho_T(x)$ of $T$ at point $x \in X$ is defined as the set of every $\lambda \in \mathbb{C}$ such that there exists a neighborhood $U_{\lambda}$ of $\lambda$ and $f : U_{\lambda} \to X$ such that $(T - \mu)f(\mu) = x \ \forall \mu \in U_{\lambda}$. The local spectrum $\sigma_T(x)$ of $T$ at $x$ is defined as $\sigma_T(x) = \mathbb{C}\setminus \rho_T(x)$. We observe that the local analytical solution of the equation given in the definition of the local resolvent will be unique if and only if $T$ has SVEP.

For all subset $F$ of $\mathbb{C}$, the local spectral space of $T$ associated to $F$ is defined by

$$X_T(F) = \{x \in X; \sigma_T(x) \subset F\}.$$ 

Obviously that $X_T(F)$ is a hyperinvariant subspace of $T$, but not necessarily
closed.
An operator $T \in L(X)$ has the property of Dunford (C) if $X_T(F)$ is closed for every closed set $F$ of $\mathbb{C}$.
We denote by $D(\lambda, r)$ the disk centered at $\lambda \in \mathbb{C}$ and has a radius $r > 0$.
We denote by $\mathcal{O}(U, X)$ the Fréchet algebra of all $X$-valued analytic functions on the open subset $U \subset \mathbb{C}$ endowed with uniform convergence on compact subsets of $U$.
An operator $T \in L(X)$ satisfies the Bishop property ($\beta$) if there exists $r > 0$ such that for any open subset $U \subset D(0, r)$ and for any sequence $(f_n)_{n=1}^{\infty} \subset \mathcal{O}(U, X)$ such that $\lim_{n \to \infty} (T - \mu)f_n(\mu) = 0$ in $\mathcal{O}(U, X)$, then $\lim_{n \to \infty} f_n(\mu) = 0$ in $\mathcal{O}(U, X)$. We denote by $\sigma_\beta(T)$ the set of $\lambda$ such that $T$ does not satisfy the property ($\beta$).

We say that $T$ satisfies the property ($\beta$) if $\sigma_\beta(T) = \emptyset$. We say that $T \in L(X)$ has the Decomposition property ($\delta$) if $T^*$ satisfies the property ($\beta$).
$T \in L(X)$ is decomposable in the sense of Foias [5] if and only if $T$ satisfies both ($\beta$) and ($\delta$). The same way we define the property ($\beta_\epsilon$). We denote by $\xi(U, X)$ the Fréchet algebra of all $X$-valued infinitely continuously differentiable on the open subset $U \subset \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of $U$ of all derivatives.
An operator $T \in L(X)$ satisfies the property ($\beta_\epsilon$) at $\lambda \in \mathbb{C}$ if there exists neighborhood $U_\lambda$ of $\lambda$ such that for all open subsets $U \subset U_\lambda$ and for any sequence $(f_n)_{n=1}^{\infty} \subset \xi(U, X)$ such as $\lim_{n \to \infty} (T - \mu)f_n(\mu) = 0$ in $\xi(U, X)$, implies $\lim_{n \to \infty} f_n(\mu) = 0$ in $\xi(U, X)$.
we denote by $\sigma_{\beta_\epsilon}(T)$ the set of $\lambda$ such that $T$ does not satisfy the property ($\beta_\epsilon$).

We say that $T$ satisfies the property ($\beta_\epsilon$) if $\sigma_{\beta_\epsilon}(T) = \emptyset$. We know that $T$ has the property ($\beta_\epsilon$) if $T$ is a subscalar. For further definitions and more details, we refer the reader to [1] and [6].
We can prove the following implications (see [5] and [6]) : 

Subscalar $\Rightarrow$ property of Bishop$(\beta)$ $\Rightarrow$ property of Dunford (C) $\Rightarrow$ SVEP.

For every closed set $F$ of $\mathbb{C}$, the global spectral subspace $\mathcal{X}_T(F)$ is defined as the set of all point $x \in X$ such that there exists an analytic function $f : \mathbb{C} \setminus F \to X$ such that $(T - \lambda)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$.
Clearly $\mathcal{X}_T(F)$ is a hyperinvariant subspace of $T$ and $\mathcal{X}_T(F) \subset X_T(F)$. Plus we will get the equality $\mathcal{X}_T(F) = X_T(F)$ on any closed set $F$ of $\mathbb{C}$ when $T$ has
SVEP, see [6, proposition 3.3.2]

Let \( T \in L(X) \) and \( S \in L(Y) \), the commutator \( C(S, T) \in L(L(X), L(Y)) \)
introducing by Colojoara and Foias in [5] is the mapping defined by
\[
C(S, T)(A) = SA - AT \quad \text{for all } A \in L(X, Y).
\]

A naturel link between the operators \( T \in L(X) \) and \( S \in L(Y) \) is provided by
the intertwining condition \( SA = AT \) for some non-zero operator \( A \in L(X, Y) \).
The iterates \( C(S, T)^n \) of the commutator are defined, in the usual fashion, for
all \( A \in L(X, Y) \) and any \( n \in \mathbb{N} \) by
\[
C(S, T)^0(A) = A \quad \text{and} \quad C(S, T)^n = C(S, T)(C(S, T)^{n-1}(A)).
\]

By induction we can be show that
\[
C(S, T)^n(A) = \sum_{k=0}^{n} C_n^k(-1)^k S^{n-k} A T^k
\]

If \( X = Y \) and \( S, T \) and \( A \) commute by pairs with each other it can be deduced that
\[
C(S, T)^n(A) = (S - T)^n A.
\]

The commutators play an important role in the theory of local spectral inclusions, see [1], [5] and [6].

Recall that the local spectral radius of the commutator \( C(S, T) \) at \( A \in L(X, Y) \) is defined as [6]
\[
\rho_{C(S, T)}(A) = \limsup_{n \to \infty} \|C(S, T)^n(A)\|^{\frac{1}{n}}.
\]

According to [6, proposition 3.4.2] we have
\[
A \mathcal{X}_T(F) \subset \mathcal{Y}_S(F + D(0, r)) \quad \text{for all closed set } F \subset \mathbb{C}.
\]

We say that the pair \( (S, T) \in L(Y) \times L(X) \) is asymptotically interwined by
the operator \( A \in L(X, Y) \) if
\[
\lim_{n \to \infty} \|C(S, T)^n(A)\|^{\frac{1}{n}} = 0.
\]

Evidently, this notion is a generalization of the ordinary intertwining condition \( SA = AT \), and also of the higher order intertwining condition
\[
C(S, T)^n(A) = 0 \quad \text{for some } n \in \mathbb{N}
\]

If \( T \) has SVEP we obtain
\[
\sigma_S(Ax) \subset \sigma_T(x) + D(0, r_{C(S, T)}(A)) \quad \text{for all } x \in X.
\]

If \( r_{C(S, T)}(A) = 0 \) we obtain \( \sigma_S(Ax) \subset \sigma_T(x) \).
The properties of local spectrum between $AT$ and $SA$

We begin with the following theorem which gives some relation between the local spectrum.

**Theorem 2.1.** Let $X,Y$ be two Banach spaces, $S,T \in L(Y,X)$ and $A,B \in L(X,Y)$ such that

$$BSA = ATB.$$

Then we have the following inclusions

1. $\sigma_{AT}(Bx) \subset \sigma_{SA}(x)$
2. $\sigma_{TA}(TBSy) \subset \sigma_{AS}(y)$.

**Proof:**

1. We assume that $\lambda \notin \sigma_{SA}(x)$. Then there exists an analytic function $f : U_\lambda \to X$ defined on some neighborhood $U_\lambda$ of $\lambda$ such that

$$(SA - \mu)f(\mu) = x \text{ pour tout } \mu \in U_\lambda.$$

Thus, $B(SA - \mu)f(\mu) = (AT - \mu)B f(\mu) = Bx$ and $\lambda \notin \sigma_{AT}(Bx)$.

2. We know that $\sigma_{TA}(Tz) \subset \sigma_{AT}(z)$ and by using (1) we get

$$\sigma_{TA}(TBSy) \subset \sigma_{AT}(BSy) \subset \sigma_{SA}(Sy) \subset \sigma_{AS}(y).$$

If $A = B$ we obtain the theorem 2.3 in [13]

**Corollary 2.1.** In particular $A = B$ we get

1. $\sigma_{AT}(Ax) \subset \sigma_{SA}(x) \subset \sigma_{AT}(Ax) \cup \{0\}$
2. $\sigma_{TA}(TASy) \subset \sigma_{AS}(y) \subset \sigma_{TA}(TASy) \cup \{0\}$

**Proof:** Applique the theorem 2.1.

**Theorem 2.2.** Let $X,Y$ be two Banach spaces, $S,T \in L(Y,X)$ and $A,B \in L(X,Y)$ such that

$$BSA = ATB.$$

Then
1. If $B$ is injective, then
$$\sigma_\beta(SA) \subset \sigma_\beta(AT)$$
In particular if $AT$ has the property $\beta$, then $SA$ shares the same property.

2. If $B$ is injective, then $\sigma_\beta(SA) \subset \sigma_\beta(AT)$. In particular if $AT$ is subscalar, then $SA$ is also subscalar.

3. If $A = B$ we will have the equality of these inclusions.

Proof:

1. Let $\lambda \notin \sigma_\beta(AT)$. On open set of $U \subset D(\lambda, r)$ of $\mathbb{C}$ and a sequence $(g_n(\mu))_{n \in \mathbb{N}}$ of $O(U, X)$ such that
$$\lim (SA - \mu)g_n(\mu) = 0$$
$$\lim (BSA - \mu B)g_n(\mu) = 0$$
$$\lim (ATB - \mu B)g_n(\mu) = 0$$
$$\lim (AT - \mu)Bg_n(\mu)) = 0$$

Since
$$\lim Bg_n(\mu) = 0$$
and as $B$ is injective, then $\lim g_n(\mu) = 0 \ \forall \mu \in U$.

Since
$$\lambda \notin \sigma_\beta(SA).$$

Similarly for the other inclusion.

2. The same way as before.

3. see theorem 2.1 in [13] ■

Corollary 2.2. Let $X,Y$ two Banach spaces, $S,T \in L(Y,X)$ and $A,B \in L(X,Y)$ such that
$$BSA = ATB.$$ If $B$ is surjective, then
$$\sigma_\beta((AT)^*) \subset \sigma_\beta((SA)^*)$$
In particular if $SA$ has the property $(\delta)$, then $AT$ has it also.
Proof: Since $B$ is surjective, then $B^*$ is injective and apply the previous theorem 2.2 we obtained the results. □

Theorem 2.3. Let $A, B \in L(X,Y)$ and $S, T \in L(Y,X)$ such that $BSA = ATB$. Then for all $d \in \mathbb{N}$ we have

1. $BR(SA - I)^d \subset R(AT - I)^d$
2. $BN(SA - I)^d \subset N(AT - I)^d$
3. $SATR(BT - I)^d \subset R(SB - I)^d$
4. $SATN(BT - I)^d \subset N(SB - I)^d$

Proof:

1. Let $y \in R(SA - I)^d$, then there exists $x \in X$ such that $(SA - I)^d(x) = y$ Whence $B(SA - I)^d = By$ without forgetting the equality $BSA = ATB$ we obtain $(AT - I)^d(Bx) = By$ and therefore $By \in R(AT - I)^d$
   In conclusion $BR(SA - I)^d \subset R(AT - I)^d$
2. Same way as in the previous property.
3. Let $y \in R(BT - I)^d$, then there exists $x \in Y$ such that $(BT - I)^d(x) = y$ whence $SAT(BT - I)^d(x) = SATy$ with the equality $BSA = ATB$ we obtain $(SB - I)^dSAT(x) = SATy$ therefore $SATy \in R(SB - I)^d$ And consequently $SATR(BT - I)^d \subset R(SB - I)^d$
4. The same way as the previous □

Consequently we have the following result, For more details see lemma 2.3 [12]

Corollary 2.3. If $A = B$, then we will obtain for all $d \in \mathbb{N}$ we have

1. $AR(SA - I)^d \subset R(AT - I)^d$
2. $AN(SA - I)^d \subset N(AT - I)^d$
3. $SATR(AT - I)^d \subset R(SA - I)^d$
4. $SATN(AT - I)^d \subset N(SA - I)^d$

Under the injectivity of $B$ the following theorem show that the SVEP holds for $AT$ and $SA$.

Theorem 2.4. Let $S, T \in L(Y,X)$ and $A, B \in L(X,Y)$ such that

$$BSA = ATB.$$ 

If $B$ is injective, then $AT$ has SVEP, implies $SA$ has it also.
Proof: Let $S, T \in L(Y,X)$ and $A, B \in L(X,Y)$ and $f : U \to \mathbb{C}$ is an analytic function on an neighborhood $U$ of $X$ such that

$$(SA - \mu)f(\mu) = 0 \text{ for all } \mu \in U$$

$$B(SA - \mu)f(\mu) = 0 \text{ for all } \mu \in U$$

$$(AT - \mu)Bf(\mu) = 0 \text{ for all } \mu \in U.$$  

Therefore $AT$ has SVEP then $Bf(\mu) = 0$ for all $\mu \in U$. Plus $B$ is injective, hence $f \equiv 0$ on $U$. Finally $AT$ has SVEP \qed

3 Commutator and permanent inclusions

In this section we establish some results concerning the commutator.

Definition 3.1. Let $A, B \in L(X,Y)$ and $S, T \in L(Y,X)$ We define the bilinear application $f_{(S,T)}$ such that

$$f_{(S,T)}(A, B) = BSA - ATB$$

We also set the linear applications $g_A$ et $g_B$ such that

$$g_A : B \to f_{(S,T)}(A, B)$$

$$g_B : A \to f_{(S,T)}(A, B).$$

Remark 1. If we consider the notations in [6], we can write $f_{(S,T)}$

$$f_{(S,T)}(A, B) = BSA - ATB = -[(AT)B - B(SA)] = -C_{(AT, SA)}(B)$$

$$f_{(S,T)}(A, B) = (BS)A - A(TB) = C_{(BS, TB)}(A)$$

So one can see that $g_A = -C_{(AT, SA)}$ and $g_B = C_{(BS, TB)}$.

Definition 3.2. The iterations of $g_A$ and $g_B$ are defined by :

$$g^n_B(A) = A, g^n_B(A) = [C_{(BS, TB)}(A)] = C_{(BS, TB)}([C_{(BS, TB)}]^{n-1})$$

$$g^n_A(B) = B, g^n_A(B) = (-1)^n[C_{(AT, SA)}(B)] = (-1)^nC_{(AT, SA)}([C_{(AT, SA)}]^{n-1}).$$

We set

$$r_{f_{(S,T)}}(A) = r_{g_B}(A) = r_{C_{(BS, TB)}}(A) = \limsup \|C_{(BS, TB)}(A)\|^\frac{1}{n} = r_B(A)$$

$$r_{f_{(S,T)}}(B) = r_{g_A}(B) = r_{C_{(AT, SA)}}(B) = \limsup \|C_{(AT, SA)}(B)\|^\frac{1}{n} = r_A(B)$$

$$r_{f_{(S,T)}}(A, B) = \max\{r_{f_{(S,T)}}(A), r_{f_{(S,T)}}(B)\} = r_f(A, B)$$
Remark 2. If $X = Y$ and $A, B, S$ and $T$ commute by pairs with each other then we deduce the following formulas

1. $g^n_B = AB^n(S - T)^n = A(B(S - T))^n$, in this case $g^n_B = 0$ if and only if $BS = BT + N$ with $N$ is nilpotent of index $n$
2. $g^n_A = BA^n(S - T)^n = B(A(S - T))^n$, in this case $g^n_A = 0$ if and only if $AS = AT + N$ with $N$ is nilpotent of index $n$
3. If in addition $B = I$, then $f_{(S,T)}(A, I) = SA - AT = C_{(S,T)}(A)$
4. If in addition $A = I$, then $f_{(S,T)}(I, B) = BS - TB = -C_{(T,S)}(B)$

Proposition 3.1. Let $A, B \in L(X, Y)$ and $S, T \in L(Y, X)$. for every closed set $F \subset \mathbb{C}$ we have

1. $AXSA(F) \subset YAT(F + D(0, r_A(B)))$.
   If in addition $SA$ has SVEP , then $BXSA(F) \subset YAT(F + D(0, r_A(B)))$.
   Therefore $\sigma_{AT}(Bx) \subset \sigma_{SA}(x) + D(0, r_A(B))$
2. $AXTB(F) \subset YBS(F + D(0, r_B(A)))$.
   If in addition $TB$ has SVEP , then $AXTB(F) \subset YBS(F + D(0, r_B(A)))$.
   Since $\sigma_{BS}(Ax) \subset \sigma_{TB}(x) + D(0, r_B(A))$.

Proof : See [2, proposition 3.4.2] ■

Corollary 3.1. If $r_B(A) = 0$ we obtain

$$\sigma_{BS}(Ax) \subset \sigma_{TB}(x)$$

and in addition $A = B$ we will get

$$\sigma_{AS}(Ax) \subset \sigma_{TA}(x).$$

Proposition 3.2. Let $A, B \in L(X, Y)$ and $S, T \in L(Y, X)$. Suppose that $SA$ has the property $(\delta)$. Then for every $B \in L(X, Y)$ and every $r \geq 0$, we have the following equivalent properties :

1. $r_A(B) \leq r$
2. $BXSA(F) \subset YAT(F + D(0, r))$ for every closed set $F \subset \mathbb{C}$

Proof : See [2, theorem 3.4.3] ■

Références

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