On the Ideal Center of a Dual Vector Lattice

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Abstract

Let $X$ be a vector lattice with order dual $X'$. In this paper, we investigate when $Orth(X')$ is an ideal center in $X'$. In [6], Toumi answers related question posed by Wickstead in [1]. This study is the dual version of the paper of Toumi in [6].

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1 Introduction

Let $X$ be an Archimedean vector lattice. $Orth(X)$ denotes the vector lattice of all orthomorphisms on $X$. $Z(X)$ denotes the sublattice of $Orth(X)$ consisting of those $\pi$ for which there is non-negative real number $\lambda$ with $-\lambda x \leq \pi(x) \leq \lambda x$ for all $x \in X^+$ (Positive cone of $X$). We say that $Z(X)$ is the ideal center of $X$. A vector lattice $X$ under an associative multiplication is said to be lattice-ordered algebra ($\ell$-algebra) whenever the multiplication makes $X$ an algebra, in addition it satisfies the following property: If $x, y \in X^+$, then $xy \in X^+$. [2]. A lattice-ordered algebra $X$ is said to be an $f$-algebra when $x \land y = 0$ implies $xz \land y = zx \land y = 0$ for each $z \in X^+$. Note that any Archimedean $f$-algebra is commutative. Moreover, squares in an $f$-algebra are positive.[3].

If $X$ is an $f$-algebra, the mapping $\rho: X \to Orth(X)$ defined by $\rho(f) = \pi_f$ where $\pi_f(g) = fg$ for all $f, g \in X$ is an algebra and lattice homomorphism. If $\rho(f) \in Z(X)$, $f$ is said to be bounded and if every element of $X$ is bounded, the $f$-algebra $X$ is said to be bounded.[5].

Let $X$ be a vector lattice and let $0 \leq v \in X$. If for every real number $\varepsilon > 0$, there exists a natural number $n_\varepsilon$ such that $|x_n - x| \leq \varepsilon v$ for all $n \geq n_\varepsilon$, the sequ-
ence \( \{x_n\}_{n \geq 0} \) in \( X \) is called \((v)\) relatively uniformly convergent to \( x \in X \) which is denoted by \( x_n \to x(v) \). That is, if \( x_n \to x(v) \) for some \( 0 \leq v \in X \), then the sequence \( \{x_n\}_{n \geq 0} \) is called (relatively) uniformly convergent to \( x \). This is denoted by \( x_n \to x(\text{r.u}) \). A vector lattice is called relatively uniformly convergent if every relatively uniform Cauchy sequence in \( X \) has a unique limit. Relatively uniform limits are unique if \( X \) is Archimedean.

The relatively uniform topology is denoted by \((\text{r.u})\) topology.

Let \( X \) be a topological algebra and \( M(X) \) be the set of all maximal two sided ideals in \( X \). The space \( M(X) \) is equipped with the \( h_k \)-topology: \( S \subset M(X) \) is closed if \( S = H(K(S)) \), where \( K(S) \) is the intersection of all ideals in \( S \) and \( H(I) = \{ M \in M(X) : I \subset M \} \) for any two-sided ideal \( I \) of \( X \). If \( X \) is also equipped with a compatible topology we consider a subset \( m(X) \) of \( M(X) \) consisting of closed ideals. We use small letters \( h \) and \( k \) to indicate \( H \) and \( K \) restricted to \( m(X) \).

A lattice norm \( \| \cdot \| \) on \( X \) is called an \( M \)-norm if \( \| x \lor y \| = \max \{ \| x \|, \| y \| \} \) for all \( x, y \in X^+ \).

Note that, for unexplained terminology and notation, we refer to the standard books [3] and [7]. Next theorem is seen in the work of Basly and Triki in [5] and [6].

**Theorem 1** If \( X \) is a relatively uniformly complete \( f \)-algebra, the following are equivalent:

i. \( X \) is bounded.

ii. Every maximal modular ring ideal in \( X \) is uniformly closed.

iii. Every maximal modular ring ideal in \( X \) is the kernel of a lattice and algebra homomorphism \( X \to R \).

iv. Every maximal modular ring ideal in \( X \) is a maximal order ideal.

**Theorem 2** [6] Consider a relatively uniformly complete unital \( f \)-algebra \( X \). Then \( M(X) = m(X) \), with respect to the relatively uniform topology, if and only if

i. Every proper finitely generated ring ideal is contained in a uniformly closed maximal ring ideal of \( X \).

ii. \( m(X) \) is compact in \( h_k \)-topology.

**Proposition 3** [6] Any maximal ring ideal of a relatively uniformly complete semiprime \( f \)-algebra \( X \) is an order ideal.

Now, we will give the dual version of the study in [6].

**Theorem 4** [6] Let \( X \) be a vector lattice and \( X' \) be order dual of \( X \). Then the following results are equivalent:

i. \( \text{Orth}(X') = Z(X') \).
On the ideal center of a dual vector lattice

ii. \( M(\text{Orth}(X')) = m(\text{Orth}(X')) \), with respect to the relatively uniform topology.

iii. Every proper finitely generated ring ideal is contained in a uniformly closed maximal ring ideal of \( \text{Orth}(X') \) and \( m(\text{Orth}(X')) \) is compact in the \( hk \)-topology.

**Proof:** i\(\Rightarrow\)ii Because \( X' \) is Dedekind complete, \( \text{Orth}(X') \) is Dedekind complete. So, \( \text{Orth}(X') \) is relatively uniformly complete. First, define the gauge function \( P(\pi) := \inf \{ \lambda \in R : -\lambda I_{X'} \leq \pi \leq \lambda I_{X'} \} \) for \( \pi \in \text{Orth}(X') \). Since \( \text{Orth}(X') = Z(X') \), \( P(\pi) \) is an \( M \)-norm. Moreover, \( (\text{Orth}(X'),P) \) is a Banach lattice. \( \text{Orth}(X') \) is a unital \( f \)-algebra under composition, so \( \text{Orth}(X') \) is a Banach unital \( f \)-algebra. Because every maximal ring ideal in \( \text{Orth}(X') \) is uniformly closed, any maximal ring ideal in \( \text{Orth}(X') \) is closed, with respect to the norm topology.

i\(\Leftrightarrow\)iii From the Theorem 2, it is proven.

ii\(\Rightarrow\)i Assume by way of contradiction that \( Z(X') \not\subseteq \text{Orth}(X') \). There exists \( \pi \in (\text{Orth}(X'))^+ \) such that \( \pi \notin Z(X') \). Suppose that \( \varphi = \pi \wedge I_{X'} \in (\text{Orth}(X'))^+ \) and \( \omega_n = \pi - (\pi \wedge n\varphi) \), for all \( n \in N \). Because of \( \pi \notin Z(X') \), we get \( \omega_n = \pi - (\pi \wedge n\varphi) > 0 \). Then \( \{\omega_n\}^d \) is proper uniformly closed order ideal of \( \text{Orth}(X') \). Moreover, we have \( \{\omega_n\}^d \subseteq \{\omega_{n+1}\}^d \), for all \( n \geq 1 \). Let \( A = \bigcup_{n \geq 1} \{\omega_n\}^d \), then \( A \) is a proper ring ideal of \( \text{Orth}(X') \). Since \( (I_{X'} - \varphi) \wedge (\pi - \varphi) = 0 \), we get \( \psi(I_{X'} - \varphi) \wedge (\pi - \varphi) = 0 \) for all \( \psi \in \text{Orth}(X') \). It follows that \( \psi(I_{X'} - \varphi) = \psi - \psi\varphi \in \{\omega_1\}^d \) and therefore, \( \psi - \psi\varphi \in A \) for all \( \psi \in \text{Orth}(X') \). Because there exists a closed maximal ring ideal \( M \) such that \( A \subseteq M \) and \( M(\text{Orth}(X')) = m(\text{Orth}(X')) \), \( M \) is relatively uniformly closed order ideal of \( \text{Orth}(X') \). So, the vector lattice quotient \( \text{Orth}(X')/M \) is Archimedean.

Let \( \pi_n = (\pi \vee n\varphi) - \pi \) for all \( n \in N \). It follows that \( \pi_n\omega_n = 0 \). Therefore, \( \pi_n \in \{\omega_n\}^d \subseteq A \). If we pass to the classes modulo \( M \), we have \( \pi_n = \overline{(\pi \vee n\varphi) - \pi} = \overline{\pi - \pi} = \overline{\pi} = \overline{\pi} \) for all \( n \geq 1 \) and so \( 0 \leq \overline{n\varphi} \leq \overline{\pi} \) for all \( n \geq 1 \). Since \( \text{Orth}(X')/M \) is Archimedean, it follows that \( \varphi \in M \). Since the relation \( \psi - \psi\varphi \in A \), we get \( \psi \in M \). Consequently we get the equation \( M = \text{Orth}(X') \) which is a contradiction. This completes the proof of the theorem.

**Corollary 5** [6] If \( X \) is Banach lattice then \( \text{Orth}(X') = Z(X') \).

**Proof:** Since \( X \) is a Banach lattice, any orthomorphism \( \pi \) on \( X' \) is continuous,[7]. On the other hand, since the modulus of \( \pi \) exists [4], it shows that \( \text{Orth}(X') \) is a Banach unital \( f \)-algebra under composition. It is clear that every maximal ring ideal in \( \text{Orth}(X') \) is closed with respect to the norm topology,[4]. So, every maximal ring ideal in \( \text{Orth}(X') \) is uniformly closed. Then \( M(\text{Orth}(X')) = m(\text{Orth}(X')) \) with respect to the relatively uniform topology. From the equivalent conditions ii\(\Rightarrow\)i in Theorem 4, we get \( \text{Orth}(X') = Z(X') \).

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References


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