Hyperidentities in \((xx)y \approx xy\) Graph Algebras
of Type \((2,0)\)

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Abstract

Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type \((2,0)\). We say that a graph \(G\) satisfies an identity \(s \approx t\) if the corresponding graph
algebra \(A(G)\) satisfies \(s \approx t\). A graph \(G = (V, E)\) is called a \((xx)y \approx xy\)
graph if the graph algebra \(A(G)\) satisfies the equation \((xx)y \approx xy\). An
identity \(s \approx t\) of terms \(s\) and \(t\) of any type \(\tau\) is called a hyperidentity
of an algebra \(A\) if whenever the operation symbols occurring in \(s\) and \(t\) are replaced
by any term operations of \(A\) of the appropriate arity, the resulting identities hold in \(A\).

In this paper we characterize \((xx)y \approx xy\) graph algebras, identities
and hyperidentities in \((xx)y \approx xy\) graph algebras.

Mathematics Subject Classification: 03C05, 05C25

Keywords: identities, hyperidentities, term, normal form term, binary
algebra, graph algebra, \((xx)y \approx xy\) graph algebra
1 Introduction

An identity $s \approx t$ of terms $s, t$ of any type $\tau$ is called a hyperidentity of an algebra $A$ if whenever the operation symbols occurring in $s$ and $t$ are replaced by any term operations of $A$ of the appropriate arity, the resulting identity holds in $A$. Hyperidentities can be defined more precisely using the concept of a hypersubstitution.

We fix a type $\tau = (n_i)_{i \in I}, n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i \in I}$, where $f_i$ is $n_i - ary$. Let $W_\tau(X)$ be the set of all terms of type $\tau$ over some fixed alphabet $X$, and let $Alg(\tau)$ be the class of all algebras of type $\tau$. Then a mapping

$$\sigma : \{f_i | i \in I\} \rightarrow W_\tau(X)$$

which assigns to every $n_i - ary$ operation symbol $f_i$ an $n_i - ary$ term will be called a hypersubstitution of type $\tau$ (for short, a hypersubstitution). By $\hat{\sigma}$ we denote the extension of the hypersubstitution $\sigma$ to a mapping

$$\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X).$$

The term $\hat{\sigma}[t]$ is defined inductively by

(i) $\hat{\sigma}[x] = x$ for any variable $x$ in the alphabet $X$, and
(ii) $\hat{\sigma}[f_i(t_1, ..., t_{n_i})] = \sigma(f_i)_{W_\tau(X)}(\hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}]).$

Here $\sigma(f_i)_{W_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra with the universe $W_\tau(X)$.

Graph algebras have been invented in [10] to obtain examples of nonfinitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set $V$ and the set of edges $E \subseteq V \times V$. Define the graph algebra $A(G)$ corresponding to $G$ with the underlying set $V \cup \{\infty\}$, where $\infty$ is a symbol outside $V$, and with two basic operations, namely a nullary operation pointing to $\infty$ and a binary one denoted by juxtaposition, given for $u, v \in V \cup \{\infty\}$ by

$$uv = \begin{cases} u, & \text{if } (u, v) \in E, \\ \infty, & \text{otherwise}. \end{cases}$$

Graph identities were characterized in [3] by using the rooted graph of a term $t$, where the vertices correspond to the variables occurring in $t$. Since on a graph algebra we have one nullary and one binary operation, $\sigma(f)$ in this case is a binary term in $W_\tau(X)$, i.e. a term built up from variables of a two-element alphabet and a binary operation symbol $f$ corresponding to the binary operation of the graph algebra.

In [9] R. Pöschel has shown that any term over the class of all graph algebras can be uniquely represented by a normal form term and that there is an algorithm to construct the normal form term to every given term $t$. 

We say that a graph $G = (V, E)$ is called $(xx)y \approx xy$ if the corresponding graph algebra $A(G)$ satisfied the equation $(xx)y \approx xy$. In this paper we characterize $(xx)y \approx xy$ graph algebras, identities and hyperidentities in $(xx)y \approx xy$ graph algebras.

2 $(xx)y \approx xy$ Graph Algebras

We begin with one more precise definition of terms of the type of graph algebras.

**Definition 2.1** The set $W_\tau(X)$ of all terms over the alphabet

$$X = \{x_1, x_2, x_3, ...\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1, 2, 3, ...$, and $\infty$ are terms;
(ii) if $t_1$ and $t_2$ are terms, then $f(t_1, t_2)$ is a term; instead of $f(t_1, t_2)$ we will write $t_1 \cdot t_2$, for short;
(iii) $W_\tau(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $W_\tau(X_2)$. The leftmost variable of a term $t$ is denoted by $L(t)$ and rightmost variable of a term $t$ is denoted by $R(t)$. A term, in which the symbol $\infty$ occurs is called a trivial term.

**Definition 2.2** To each non-trivial term $t$ of type $\tau = (2, 0)$ one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in $t$, and where the edge set $E(t)$ is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1 \cdot t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}.$$
when \( t = t_1t_2 \) is a compound term and \( L(t_1), L(t_2) \) are the leftmost variables in \( t_1 \) and \( t_2 \), respectively.

\( L(t) \) is called the root of the graph \( G(t) \), and the pair \((G(t), L(t))\) is the rooted graph corresponding to \( t \). Formally, to every trivial term \( t \) we assign the empty graph \( \phi \).

**Definition 2.3** We say that a graph \( G = (V, E) \) satisfies an identity \( s \approx t \) if the corresponding graph algebra \( A(G) \) satisfies \( s \approx t \) (i.e. we have \( s = t \) for every assignment \( V(s) \cup V(t) \to V \cup \{\infty\} \)), and in this case, we write \( G \models s \approx t \).

**Definition 2.4** Let \( G = (V, E) \) and \( G' = (V', E') \) be graphs. A homomorphism \( h \) from \( G \) into \( G' \) is a mapping \( h : V \to V' \) carrying edges to edges, that is, for which \((u, v) \in E \implies (h(u), h(v)) \in E' \).

In [3] it was proved:

**Proposition 2.1** Let \( s \) and \( t \) be non-trivial terms from \( W_{\tau}(X) \) with variables \( V(s) = V(t) = \{x_0, x_1, ..., x_n\} \) and \( L(s) = L(t) \). Then a graph \( G = (V, E) \) satisfies \( s \approx t \) if and only if the graph algebra \( A(G) \) has the following property:

A mapping \( h : V(s) \to V \) is a homomorphism from \( G(s) \) into \( G' \) iff it is a homomorphism from \( G(t) \) into \( G \).

Proposition 2.1 gives a method to check whether a graph \( G = (V, E) \) satisfies the equation \( s \approx t \). Hence, we can check whether a graph \( G = (V, E) \) has a \((xx)y \approx xy \) graph algebra.

Then we have:

**Proposition 2.2** Let \( G = (V, E) \) be a graph. Then \( G = (V, E) \) has a \((xx)y \approx xy \) graph algebra if and only if for any \( a, b \in V \), if \((a, b) \in E \), then \((a, a) \in E \).

**Proof.** Suppose that \( G = (V, E) \) has a \((xx)y \approx xy \) graph algebra. Let \( a, b \in V \) and \((a, b) \in E \). We will show that \((a, a) \in E \). Let \( s \) and \( t \) be terms such that \( s = (xx)y \) and \( t = xy \). Let \( h : V(t) \to V \) be the restriction of an evaluation of the variables such that \( h(x) = a \) and \( h(y) = b \). We see that \( h \) is a homomorphism from \( G(t) \) into \( G \). Since \( G \) has a \((xx)y = xy \) graph algebra thus by Proposition 2.1, we have \( h \) is a homomorphism from \( G(s) \) into \( G \). Since \((x, x) \in E(s) \), we get \((h(x), h(x)) = (a, a) \in E \).

Conversely, suppose that \( G = (V, E) \) is a graph such that for any \( a, b \in V \), if \((a, b) \in E \), then \((a, a) \in E \). Let \( s \) and \( t \) be terms such that \( s = (xx)y \) and \( t = xy \). Clearly that, if \( h : V(s) \to V \) is a homomorphism from \( G(s) \) into \( G \), then it is a homomorphism from \( G(t) \) into \( G \). Now suppose that \( h : V(t) \to V \) is a homomorphism from \( G(t) \) into \( G \). Since \((x, y) \in E(t) \), we have \((h(x), h(y)) \in E \). By assumption, we get \((h(x), h(x)) \in E \). Hence \( h \) is a
homomorphism from \( G(s) \) into \( G \). By Proposition 2.1, we have \( A(G) \) satisfies \( s \approx t \).

From Proposition 2.2 we see that graphs which have \((xx)y \approx xy\) graph algebras are the following graphs:

\[
\begin{array}{cccccccc}
G_1 & G_2 & G_3 & G_4 & G_5 & G_6 & G_7 & G_8 \\
\end{array}
\]

and all graphs such that every induced subgraph with at most two vertices is one of these graphs.

## 3 Identities in \((xx)y \approx xy\) Graph Algebras

Graph identities were characterized in [3] by the following proposition:

**Proposition 3.1** A non-trivial equation \( s \approx t \) is an identity in the class of all graph algebras iff either both terms \( s \) and \( t \) are trivial or none of them is trivial, \( G(s) = G(t) \) and \( L(s) = L(t) \).

Further it was proved.

**Proposition 3.2** Let \( G = (V,E) \) be a graph and let \( h : X \cup \{\infty\} \rightarrow V \cup \{\infty\} \) be an evaluation of the variables such that \( h(\infty) = \infty \). Consider the canonical extension of \( h \) to the set of all terms. Then there holds: if \( t \) is a trivial term then \( h(t) = \infty \). Otherwise, if \( h : G(t) \rightarrow G \) is a homomorphism of graphs, then \( h(t) = h(L(t)) \), and if \( h \) is not a homomorphism of graphs, then \( h(t) = \infty \).

Clearly, if \( s \) and \( t \) are trivial, then \( s \approx t \) is an identity in the class of all \((xx)y \approx xy\) graph algebras and \( x \approx x \ (x \in X) \) is an identity in the class of all \((xx)y \approx xy\) graph algebras too. So we consider the case that \( s \) and \( t \) are non-trivial and different from variables. Then all identities in the class of \((xx)y \approx xy\) graph algebras are characterized by the following theorem:

**Theorem 3.1** Let \( s \) and \( t \) be non-trivial terms and let \( x_0 = L(s) \). Then \( s \approx t \) is an identity in the class of all \((xx)y \approx xy\) graph algebras if and only if the following conditions are satisfied:

(i) \( L(s) = L(t) \),

(ii) \( V(s) = V(t) \),

(iii) for any vertices \( x, y \in V(s) \) with \( x \neq y \), \( (x, y) \in E(s) \) if and only if \( (x, y) \in E(t) \),
(iv) for any vertex \( x \in V(s) \) such that \( (x, y) \notin E(s) \) for all \( y \in V(s) \) with \( y \neq x \), \( (x, x) \in E(s) \) if and only if \( (x, x) \in E(t) \).

Proof. Suppose that \( s \approx t \) is an identity in the class of all \((xx)y \approx xy\) graph algebras. Since any complete graph has a \((xx)y \approx xy\) graph algebra, it follows that \( L(s) = L(t) \) and \( V(s) = V(t) \).

Let \( x, y \in V(s) \) with \( x \neq y \). Suppose that \( (x, y) \in E(s) \) but \( (x, y) \notin E(t) \). Consider the graph \( G = (V, E) \) which \( V = V(t) \), \( E = E(t) \cup \{(u, u) | u \in V(t)\} \). By Proposition 2.2, we have \( G \) has a \((xx)y \approx xy\) graph algebras. Let \( h : V(t) \longrightarrow V \) be the restriction of an identity evaluation function of the variables. We see that \( h(s) = \infty \) and \( h(t) = x_0 \). Hence \( A(G) \) does not satisfy \( s \approx t \). By the same way, if \( (x, y) \in E(t) \) but \( (x, y) \notin E(s) \), then we can prove that \( A(G) \) does not satisfy \( s \approx t \).

Suppose that, there exists \( x \in V(s) \) such that \( (x, y) \notin E(s) \) for all \( y \in V(s) \) which \( y \neq x \) and suppose that \( (x, x) \in E(s) \) but \( (x, x) \notin E(t) \). Consider the graph \( G = (V, E) \) which \( V = \{0, 1\} \), \( E = \{(0, 0), (0, 1)\} \). By Proposition 2.2, we see that \( G \) has a \((xx)y \approx xy\) graph algebra. Let \( h : V(s) \longrightarrow V \) be the restriction of an evaluation of the variables such that \( h(x) = 1 \) and \( h(z) = 0 \) for all other \( z \in V(s) \). We see that \( h(s) = \infty \) and \( h(t) = h(L(t)) \). Hence \( A(G) \) does not satisfy \( s \approx t \).

Conversely, suppose that \( s \approx t \) and \( t \) are non-trivial terms satisfying (i), (ii), (iii) and (iv). Let \( G = (V, E) \) has a \((xx)y \approx xy\) graph algebra. Let \( h : V(s) \longrightarrow V \cup \{\infty\} \) be the restriction of the variables. Suppose that \( h \) is a homomorphism from \( G(s) \) into \( G \) and let \( (x, y) \in E(t) \). If \( x \neq y \), then by (iii), we have \( (x, y) \in E(s) \). Hence \( (h(x), h(y)) \in E \). If \( x = y \), then \( (x, x) \in E(t) \). If \( (x, z) \in E(s) \) for some \( z \in V(s) \) which \( z \neq x \), then \( (h(x), h(z)) \in E \). Since \( G \) has a \((xx)y \approx xy\) graph algebra, we get \( (h(x), h(x)) \in E \). If \( (x, z) \notin E(s) \) for all \( z \in V(s) \) which \( z \neq x \), then by (iv), we have \( (x, x) \in E(s) \). We get \( (h(x), h(x)) \in E \). Therefore \( h \) is a homomorphism from \( G(t) \) into \( G \). By the same way, if \( h \) is a homomorphism from \( G(t) \) into \( G \), then it is a homomorphism from \( G(s) \) into \( G \). Hence by Proposition 2.1, we get \( A(G) \) satisfies \( s \approx t \).

4 Hyperidentities in \((xx)y \approx xy\) Graph Algebras

Let \( G' \) be the classes of all \((xx)y \approx xy\) graph algebras and let \( IdG' \) be the set of all identities satisfied in \( G' \). Now we want to make precise the concept of a hypersubstitution for graph algebras.

Definition 4.1 A mapping \( \sigma : \{f, \infty\} \rightarrow W_r(X_2) \), where \( f \) is the operation symbol corresponding to the binary operation of a graph algebra is called graph hypersubstitution if \( \sigma(\infty) = \infty \) and \( \sigma(f) = s \in W_r(X_2) \). The graph hypersubstitution with \( \sigma(f) = s \) is denoted by \( \sigma_s \).
**Definition 4.2** An identity $s \approx t$ is a \((xx)y \approx xy\) graph hyperidentity iff for all graph hypersubstitutions $\sigma$, the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in $G'$.

If we want to check that an identity $s \approx t$ is a hyperidentity in $G'$ we can restrict ourselves to a (small) subset of $\text{Hyp}G$ - the set of all graph hypersubstitutions.

In [4] the following relation between hypersubstitutions was defined:

**Definition 4.3** Two graph hypersubstitutions $\sigma_1, \sigma_2$ are called $G'$-equivalent iff $\sigma_1(f) \approx \sigma_2(f)$ is an identity in $G'$. In this case we write $\sigma_1 \sim_{G'} \sigma_2$.

In [2] (see also [4]) the following lemma was proved:

**Lemma 4.1** If $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdG'$ and $\sigma_1 \sim_{G'} \sigma_2$ then, $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdG'$.

Therefore, it is enough to consider the quotient set $\text{Hyp}G/\sim_{G'}$.

In [9] it was shown that any non-trivial term $t$ over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term $t$. Now, we want to describe how to construct the normal form term. Let $t$ be a non-trivial term. The normal form term of $t$ is the term $NF(t)$ constructed by the following algorithm:

(i) Construct $G(t) = (V(t), E(t))$.

(ii) Construct for every $x \in V(t)$ the list $l_x = (x_{i_1}, ..., x_{i_{k(x)}})$ of all out-neighbors (i.e. $(x, x_{i_j}) \in E(t), 1 \leq j \leq k(x)$) ordered by increasing indices $i_1 \leq ... \leq i_{k(x)}$ and let $s_x$ be the term $((x x_{i_1}) x_{i_2}) ... x_{i_{k(x)}}$.

(iii) Starting with $x := L(t), Z := V(t), s := L(t)$, choose the variable $x_i \in Z \cap V(s)$ with the least index $i$, substitute the first occurrence of $x_i$ by the term $s_{x_i}$, denote the resulting term again by $s$ and put $Z := Z \setminus \{x_i\}$. While $Z \neq \emptyset$ continue this procedure. The resulting term is the normal form $NF(t)$.

The algorithm stops after a finite number of steps, since $G(t)$ is a rooted graph. Without difficulties one shows $G(NF(t)) = G(t), L(NF(t)) = L(t)$.

In [1] the following definition was given:

**Definition 4.4** The graph hypersubstitution $\sigma_{NF(t)}$, is called normal form graph hypersubstitution. Here $NF(t)$ is the normal form of the binary term $t$.

Since for any binary term $t$ the rooted graphs of $t$ and $NF(t)$ are the same, we have $t \approx NF(t) \in IdG'$. Then for any graph hypersubstitution $\sigma_t$ with $\sigma_t(f) = t \in W_x(X_2)$, one obtains $\sigma_t \sim_{G'} \sigma_{NF(t)}$.

In [1] all rooted graphs with at most two vertices were considered. Then we formed the corresponding binary terms and used the algorithm to construct normal form terms. The result is given in the following table.
By Theorem 3.1, we have the following relations:

(i) $\sigma_0 \sim_{G'} \sigma_6$,
(ii) $\sigma_5 \sim_{G'} \sigma_7$,
(iii) $\sigma_8 \sim_{G'} \sigma_{10}$,
(iv) $\sigma_9 \sim_{G'} \sigma_{11}$,
(v) $\sigma_{12} \sim_{G'} \sigma_{14} \sim_{G'} \sigma_{16} \sim_{G'} \sigma_{18}$,
(vi) $\sigma_{13} \sim_{G'} \sigma_{15} \sim_{G'} \sigma_{17} \sim_{G'} \sigma_{19}$.

Let $M_{G'}$ be the set of all normal form graph hypersubstitutions in $G'$. Then we get,

$$M_{G'} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_8, \sigma_9, \sigma_{12}, \sigma_{13}\}.$$

We defined the product of two normal form graph hypersubstitutions in $M_{G'}$ as follows.

**Definition 4.5** The product $\sigma_{1N} \circ_N \sigma_{2N}$ of two normal form graph hypersubstitutions is defined by $(\sigma_{1N} \circ_N \sigma_{2N})(f) = NF(\hat{\sigma}_{1N}[\sigma_{2N}(f)])$.

The following table gives the multiplication of elements in $M_{G'}$.

<table>
<thead>
<tr>
<th>normal form term</th>
<th>graph hypers</th>
<th>normal form term</th>
<th>graph hypers</th>
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<tbody>
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<td>$x_1 x_2$</td>
<td>$\sigma_0$</td>
<td>$x_1$</td>
<td>$\sigma_1$</td>
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<tr>
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<td>$\sigma_2$</td>
<td>$x_1 x_1$</td>
<td>$\sigma_3$</td>
</tr>
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<td>$x_2 x_1$</td>
<td>$\sigma_5$</td>
</tr>
<tr>
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<td>$(x_2 x_1) x_2$</td>
<td>$\sigma_7$</td>
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<tr>
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<td>$x_2 (x_1 x_1)$</td>
<td>$\sigma_9$</td>
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<tr>
<td>$(x_1 x_1) (x_2 x_2)$</td>
<td>$\sigma_{10}$</td>
<td>$(x_2 (x_1 x_1)) x_2$</td>
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<td>$\sigma_{12}$</td>
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<td>$\sigma_{16}$</td>
<td>$(x_2 (x_1 x_2)) x_2$</td>
<td>$\sigma_{17}$</td>
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<td>$(x_1 x_1) ((x_2 x_1) x_2)$</td>
<td>$\sigma_{18}$</td>
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</tr>
</tbody>
</table>
In [1] the concept of a leftmost normal form graph hypersubstitution was defined.

**Definition 4.6** A graph hypersubstitution \( \sigma \) is called *leftmost* hypersubstitution if 
\[
L(\sigma(f)) = x_1.
\]

The set \( M_{L(G')} \) of all leftmost normal form graph hypersubstitutions in \( M_{G'} \) contains exactly the following elements;

\[
M_{L(G')} = \{ \sigma_0, \sigma_1, \sigma_3, \sigma_8, \sigma_{12} \}.
\]

In [5] the concept of a proper hypersubstitution of a class of algebras was introduced.

**Definition 4.7** A hypersubstitution \( \sigma \) is called *proper with respect to a class \( K \) of algebras* if \( \hat{\sigma}[s] \approx \hat{\sigma}[t] \in \text{Id} K \) for all \( s \approx t \in \text{Id} K \).

A graph hypersubstitution with the property that \( \sigma(f) \) contains both variables \( x_1 \) and \( x_2 \) is called *regular*. It is easy to check that the set of all regular graph hypersubstitutions forms a groupoid \( M_{\text{reg}} \).

We want to prove that \( \{ \sigma_0, \sigma_8, \sigma_{12} \} \) is the set of all proper graph hypersubstitutions with respect to \( G' \).

In [1] the following lemma was proved.

**Lemma 4.2** For each non-trivial term \( s, (s \neq x \in X) \) and for all \( u, v \in X \), we have

\[
E(\hat{\sigma}_8[s]) = E(s) \cup \{(v,v) | (u,v) \in E(s)\},
\]

\[
E(\hat{\sigma}_{12}[s]) = E(s) \cup \{(v,u) | (u,v) \in E(s)\}.
\]

Then we obtain:
Theorem 4.1 \( \{\sigma_0, \sigma_8, \sigma_{12}\} \) is the set of all proper graph hypersubstitution with respect to the class \( \mathcal{G}' \) of \( (xx)y \approx xy \) graph algebras.

Proof. If \( s \approx t \in Id\mathcal{G}' \) and \( s, t \) are trivial terms, then \( \hat{\sigma}_8[s], \hat{\sigma}_{12}[s], \hat{\sigma}_8[t] \) and \( \hat{\sigma}_{12}[t] \) are also trivial terms and thus \( \hat{\sigma}_8[s] \approx \hat{\sigma}_8[t], \hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in Id\mathcal{G}' \). In the same manner, we see that \( \hat{\sigma}_8[s] \approx \hat{\sigma}_8[t], \hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in Id\mathcal{G}' \), if \( s = t = x \).

Now, assume that \( s \) and \( t \) are non-trivial terms, different from variables, and \( s \approx t \in Id\mathcal{G}' \). Then (i) – (iv) of Theorem 3.1 hold.

If \( |V(s)| = 1 \), then \( G(s) \) and \( G(t) \) are loops. Thus \( \hat{\sigma}_8[s], \hat{\sigma}_8[t], \hat{\sigma}_{12}[s] \) and \( \hat{\sigma}_{12}[t] \) are loops too. We have that \( \hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{G}' \) and \( \hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in Id\mathcal{G}' \).

Suppose that \( |V(s)| \geq 2 \).

For \( \sigma_8 \), we obtain:

\[
L(\hat{\sigma}_8[s]) = L(s) = L(t) = L(\hat{\sigma}_8[t]).
\]

Since \( \sigma_8 \) is regular, we have;

\[
V(\hat{\sigma}_8[s]) = V(s) = V(t) = V(\hat{\sigma}_8[t]).
\]

By Lemma 4.2, we get;

\[
E(\hat{\sigma}_8[s]) = E(s) \cup \{(v, v) \mid (u, v) \in E(s)\},
\]

\[
E(\hat{\sigma}_8[t]) = E(t) \cup \{(v, v) \mid (u, v) \in E(t)\}.
\]

Let \( (x, y) \in E(\hat{\sigma}_8[s]) \) which \( x \neq y \). We have that \( (x, y) \in E(s) \). Hence \( (x, y) \in E(\hat{\sigma}_8[t]) \). By the same way, we can prove that if \( (x, y) \in E(\hat{\sigma}_8[t]) \) which \( x \neq y \), then \( (x, y) \in E(\hat{\sigma}_8[s]) \).

Let \( x \in V(\hat{\sigma}_8[s]) \) such that \( (x, z) \notin E(\hat{\sigma}_8[s]) \) for all \( z \in V(\hat{\sigma}_8[s]) \) which \( z \neq x \). Since \( |V(s)| \geq 2 \) thus \( (w, x) \in E(\hat{\sigma}_8[s]) \) for some \( w \in V(\hat{\sigma}_8[s]) \) which \( w \neq x \). Hence \( (w, x) \in E(s) \) and \( (w, x) \in E(t) \). We have that \( (x, x) \in E(\hat{\sigma}_8[s]) \) and \( (x, x) \in E(\hat{\sigma}_8[t]) \).

By Theorem 3.1, we have that \( \hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{G}' \).

For \( \sigma_{12} \), we obtain:

\[
L(\hat{\sigma}_{12}[s]) = L(s) = L(t) = L(\hat{\sigma}_{12}[t]).
\]

Since \( \sigma_{12} \) is regular, we have;

\[
V(\hat{\sigma}_{12}[s]) = V(s) = V(t) = V(\hat{\sigma}_{12}[t]).
\]

By Lemma 4.2, we get;

\[
E(\hat{\sigma}_{12}[s]) = E(s) \cup \{(v, u) \mid (u, v) \in E(s)\},
\]

\[
E(\hat{\sigma}_{12}[t]) = E(t) \cup \{(v, u) \mid (u, v) \in E(t)\}.
\]
Let \((x, y) \in E(\hat{\sigma}_{12}[s])\) which \(x \neq y\). If \((x, y) \in E(s)\), then \((x, y) \in E(t)\). We have that \((x, y) \in E(\hat{\sigma}_{12}[t])\) if \((x, y) \notin E(s)\), then \((y, x) \in E(\hat{\sigma}_{12}[t])\). We have that 
\((y, x) \in E(t)\) and so \((x, y) \in E(\hat{\sigma}_{12}[t])\). By the same way, we can prove that if 
\((x, y) \in E(\hat{\sigma}_{12}[t])\) which \(x \neq y\), then \((x, y) \in E(\hat{\sigma}_{12}[s])\).

Since \(|V(s)| \geq 2\), if \(x \in V(\hat{\sigma}_{12}[s])\), then there exists \(z \in V(\hat{\sigma}_{12}[s])\) such that \(z \neq x\) and \((x, z) \in E(\hat{\sigma}_{12}[s])\). Hence \(\hat{\sigma}_{12}[s]\) and \(\hat{\sigma}_{12}[t]\) satisfy (iv) of Theorem 3.1.

By Theorem 3.1, we have that \(\hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in IdG\).

For any \(\sigma \notin \{\sigma_0, \sigma_8, \sigma_{12}\}\), we give an identity \(s \approx t\) in \(G\) such that \(\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin IdG\). Clearly, if \(s\) and \(t\) are trivial terms with different leftmost and different rightmost, then \(\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin IdG\), \(\hat{\sigma}_3[\sigma] \approx \hat{\sigma}_5[\sigma] \notin IdG\) and \(\hat{\sigma}_2[\sigma] \approx \hat{\sigma}_4[\sigma] \notin IdG\).

Now, let \(s = x_1((x_2x_1)x_2)\), \(t = x_1((x_2x_1)x_2)\). By Theorem 3.1, we get \(\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdG\). If \(\sigma \in \{\sigma_5, \sigma_9, \sigma_{13}\}\), then \(L(\sigma(f)) = x_2\). We see that \(L(\hat{\sigma}[s]) = x_1\) and \(L(\hat{\sigma}[t]) = x_2\) for \(\sigma \in \{\sigma_5, \sigma_9, \sigma_{13}\}\). Thus \(\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin IdG\).

Now, we apply our results to characterize all hyperidentities in the class of all \((xx)y \approx xy\) graph algebras. Clearly, if \(s\) and \(t\) are trivial terms, then \(s \approx t\) is a hyperidentity in \(G\) if and only if they have the same leftmost and the same rightmost and \(x \approx x, x \in X\) is a hyperidentity in \(G\) too. So we consider the case that \(s\) and \(t\) are non-trivial and different from variables.

In [1] the concept of a dual term \(s^d\) of the non-trivial term \(s\) was defined in the following way:

If \(s = x \in X\), then \(x^d = x\), if \(s = t_1t_2\), then \(s^d = t_2^dt_1^d\). The dual term \(s^d\)

can be obtained by application of the graph hypersubstitution \(\sigma_5, \hat{\sigma}_5[s] = s^d\).

**Theorem 4.2** An identity \(s \approx t\) in \(G\), where \(s, t\) are non-trivial and \(s \neq x, t \neq x\), is a hyperidentity in \(G\) if and only if the dual equation \(s^d \approx t^d\) is also an identity in \(G\).

**Proof.** If \(s \approx t\) is a hyperidentity in \(G\), then \(\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]\) is an identity in \(G\), i.e., \(s^d \approx t^d\) is an identity in \(G\). Conversely, assume that \(s \approx t\) is an identity in \(G\) and that \(s^d \approx t^d\) is an identity in \(G\) too. We have to prove that \(s \approx t\) is closed under all graph hypersubstitutions from \(M_G\).

If \(\sigma \in \{\sigma_0, \sigma_8, \sigma_{12}\}\), then \(\sigma\) is a proper and we get that \(\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdG\).

By assumption, \(\hat{\sigma}_5[s] = s^d \approx t^d = \hat{\sigma}_5[t]\) is an identity in \(G\).

For \(\sigma_1, \sigma_2, \sigma_3\) and \(\sigma_4\), we have \(\hat{\sigma}_1[s] = L(s) = L(t) = \hat{\sigma}_1[t], \hat{\sigma}_2[s] = L(s^d) = L(t^d) = \hat{\sigma}_2[t], \hat{\sigma}_3[s] = L(s)L(s) = L(t)L(t) = \hat{\sigma}_3[t]\) and \(\hat{\sigma}_4[s] = L(s^d)L(s^d) = L(t^d)L(t^d) = \hat{\sigma}_4[t]\).

Because of \(\sigma_8 \circ \sigma_5 = \sigma_9, \sigma_{12} \circ \sigma_5 = \sigma_{13}\) and \(\hat{\sigma}_5[\sigma_5[t']] = \hat{\sigma}_9[\sigma_5[t']]\), \(\hat{\sigma}_{12}[\hat{\sigma}_5[t']] = \hat{\sigma}_{13}[\hat{\sigma}_5[t']]\) for all \(t' \in W_7(X)\), we have that \(\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t]\) and \(\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]\) are an identity in \(G\). ■
References


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