The Practical Formulas for Differentiation of Integral Transforms

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Abstract

We have researched the practical formulas for differentiation of integral transforms used for managing differential equations with variable coefficients. The checked transforms are Laplace, Sumudu and Elzaki, and the proposed formulas can be applied to almost every equations.

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1 Introduction

There is no universal way to find the solutions for differential equations with variable coefficients. In advanced researches, we have pursued a subject on differential equations with variable coefficients[2-3, 8-9, 15-16], and many other
transforms have been pursued in order to solve integral equations and apply in partial differential equations[6, 11, 13]. In this connection, we would like to propose the practical formulas for differentiation of integral transforms used for managing differential equations with variable coefficients. Although the proposed formulas is not difficult and simple, it is practical for the use of integral transforms in order to find the solution of differential equations with variable coefficients. In this article, the researched transforms are Laplace’s, Elzaki’s and the Sumudu, and the purpose of this research is to give easy method to find solutions for differential equations with variable coefficients and for partial differential equations, by using integral transforms. To begin with, let us see the definition of Sumudu/Elzaki transform. If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform is the integral of $f(t)$ times $e^{-st}$ from $t=0$ to $\infty$. It is a function of $s$, and is defined by $\mathcal{L}(f)$; thus

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st}f(t)dt$$

provided the integral of $f(t)$ exists[10]. In the above equation, if the kernel be changed to

$$\frac{1}{u}e^{-t/u}/ue^{-t/u},$$

we call Sumudu/Elzaki transform, respectively. The detailed contents in Sumudu/Elzaki transform can be found in [1,7,14]/[4-5], respectively.

Let $E[f(t)] = T(u)$ and let $T^{(n)}(u)$ is the Elzaki transform of the derivative of $f(t)$. Elzaki showed that

$$T^{(n)}(u) = \frac{T(u)}{u^n} - \sum_{k=0}^{n-1} u^{2-n+k} f^{(k)}(0)$$

for $n \geq 1$ and

$$E[tf(t)] = u^2 \frac{dT(u)}{du} - uT(u)$$

for $E[f(t)] = T(u)$. From the definition and simple calculations, we should obtain the following results.

1) $E[y'(t)] = T(u)/u - uy(0)$
2) $E[y''(t)] = T(u)/u^2 - y(0) - uy'(0)$
3) $E[ty'(t)] = u^2 \frac{d}{du}[T(u)/u - uy(0)] - u[T(u)/u - uy(0)]$
4) $E[t^2y'(t)] = u^4 \frac{d^2}{du^2}[T(u)/u - uy(0)]$
5) $E[ty''(t)] = u^2 \frac{d}{du}[T(u)/u^2 - y(0) - uy'(0)]$
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\[-u[T(u)/u^2 - y(0) - uy'(0)]\]

6) \(E[t^2 y''(t)] = u^4 \frac{d^2}{du^2} [T(u)/u^2 - y(0) - uy'(0)].\)

7) \(E[t^2 y(t)] = u^4 \frac{d^2}{du^2} T(u)\)

for \(E(f(t)) = T(u) \) [3-5, 9].

However, these formulas are hard to apply to given equations directly because of the complexity of calculation, and so, we would like to propose these formulas as simpler forms.

2 The practical formulas for differentiation of integral transforms

We would like to propose an automatous tool for differentiation of integral transforms used for managing differential equation with variable coefficients. It is well known fact that if a function \(f(t)\) satisfies the conditions of existence of Laplace transform, then its derivative satisfies

\[\mathcal{L}(ty') = -Y - s \frac{dY}{ds}\]

and

\[\mathcal{L}(ty'') = -2sY - s^2 \frac{dY}{ds} + y(0)\]

for \(Y = \mathcal{L}(y)\). Since the forms of \(ty'\), \(ty''\) and \(t^2y''\) are frequently used, we would like to make those to formulas. Without a repetitive mention, we will use symbols \(\mathcal{L}(y)/E(f)/S(f)\) as the Laplace/Elzaki/Sumudu transform, respectively.

On the other hand, Watugula[14] proposed the Sumudu transform and showed that

\[S[tf(t)] = u \frac{d}{du} [uG(u)],\]

where \(S[f(t)] = G(u)\). It is clear that this equality can be rewritten as

\[S[tf(t)] = u \frac{d}{du} [uG(u)] = uG(u) + u^2 G'(u) = 2uG(u) - uy(0) \quad (*)\]

because of

\[G'(u) = \frac{G(u) - y(0)}{u}\]

for \(y = f(t)\).

Using this interaction formula, we would like to draw the results of (7) to (9).
Theorem 2.1 Formulas for differentiation of integral transforms;

(1) \( \mathcal{L}(ty') = -(s^2 + 1)Y + sy(0) \) for \( Y = \mathcal{L}(y) \).
(2) \( \mathcal{L}(ty'') = -(s^3 + 2s)Y + (s^2 + 1)y(0) \).
(3) \( \mathcal{L}(t^2y'') = (s^4 + 5s^2 + 2)Y - (s^3 + 4s)y(0) \).
(4) \( T(ty') = -T(u) - u^2 y(0) \) for \( E[f(t)] = T(u) \).
(5) \( T(ty'') = -\frac{2}{u} T(u) \).
(6) \( T(t^2y'') = 3T(u) + 3u^2 y(0) - u^3 y'(0) \).
(7) \( S(ty') = 2G(u) - 2y(0) - uy'(0) \) for \( S[f(t)] = G(u) \).
(8) \( S(ty'') = \frac{2}{u} G(u) - \frac{2}{u} y(0) - 2y'(0) - uy''(0) \).
(9) \( S(t^2y'') = 2G(u) - 2y(0) - 2uy'(0) - 2u^2 y''(0) \).

Proof. (1) Note that \( Y'' = s^2 Y - sy(0) - y'(0) \) and \( Y' = sY - y(0) \) for \( Y = \mathcal{L}(y) \).

\[
\mathcal{L}(ty') = -Y - s \frac{dY}{ds} = -Y - s(sY - y(0)) = -(s^2 + 1)Y + sy(0)
\]

for \( Y = \mathcal{L}(y) \).

\[
(2) \mathcal{L}(ty'') = - \frac{d}{ds} (\mathcal{L}(y'')) = - \frac{d}{ds} \left\{ s^2 Y - sy(0) - y'(0) \right\}
= -2sY - s^2 \frac{dY}{ds} + y(0) = -2sY - s^2(sY - y(0)) + y(0)
= -(s^3 + 2s)Y + (s^2 + 1)y(0). \tag{**}
\]

(3) \( \mathcal{L}(t^2y'') = - \frac{d}{ds} (\mathcal{L}(ty'')) = - \frac{d}{ds} \left\{ -(s^3 + 2s)Y + (s^2 + 1)y(0) \right\}
= (3s^2 + 2)Y + (s^3 + 2s) \frac{dY}{ds} - 2sy(0)
= (3s^2 + 2)Y + (s^3 + 2s)(sY - y(0)) - 2sy(0)
= (s^4 + 5s^2 + 2)Y - (s^3 + 4s)y(0) \]

(4) \( T(ty') = u^2 \frac{d}{du} [T(u)/u - uy(0)] - u[T(u)/u - uy(0)] \)
\[
= u^2 \left[ \frac{T'(u)u - T(u)}{u^2} - y(0) \right] - T(u) + u^2 y(0)
= uT'(u) - 2T(u).
\]

Since \( T'(u) = T(u)/u - uy(0) \), we have
\[
T(ty') = u[T(u)/u - uy(0)] - 2T(u) = -T(u) - u^2 y(0)
\]
for $E[f(t)] = T(u)$.

\begin{align*}
(5)\ T(ty'') &= u^2 \frac{d}{du} \left[ \frac{T(u)}{u^2} - y(0) - uy'(0) \right] \\
&\quad - u \left[ \frac{T(u)}{u^2} - y(0) - uy'(0) \right] \\
&= u^2 \left[ \frac{T'(u)u^2 - T(u)2u}{u^4} - y'(0) \right] - \frac{T(u)}{u} + uy(0) + u^2 y'(0) \\
&= T'(u) - \frac{2T(u)}{u} - u^2 y'(0) - \frac{T(u)}{u} + uy(0) + u^2 y'(0) \\
&= T'(u) - \frac{3}{u} T(u) + uy(0).
\end{align*}

Since $T'(u) = T(u)/u - uy(0)$, we have

\begin{align*}
T(ty'') &= - \frac{2}{u} T(u).
\end{align*}

\begin{align*}
(6)\ T(t^2 y'') &= u^4 \frac{d^2}{du^2} \left[ \frac{T(u)}{u^2} - y(0) - uy'(0) \right] \\
&= u^4 \frac{d}{du} \left[ \frac{T'(u)u - 2T(u)}{u^3} - y'(0) \right] \\
&= u^4 \frac{d}{du} \left[ \frac{T'(u)}{u^2} - \frac{2T(u)}{u^3} - y'(0) \right].
\end{align*}

Differentiating this equality and organizing, we have

\begin{align*}
T(t^2 y'') &= u^2 T''(u) - 4u T'(u) + 6T(u).
\end{align*}

Since

\begin{align*}
T''(u) &= T(u)/u^2 - y(0) - uy'(0)
\end{align*}

and

\begin{align*}
T'(u) &= T(u)/u - uy(0),
\end{align*}

we have

\begin{align*}
T(t^2 y'') &= u^2 \{ T(u)/u^2 - y(0) - uy'(0) \} \\
&\quad - 4u \{ T(u)/u - uy(0) \} + 6T(u) \\
&= 3T(u) + 3u^2 y(0) - u^3 y'(0)
\end{align*}

for $E[f(t)] = T(u)$.

(7) Kilicman[7] showed that the Sumudu transform of the $f^{(n)}(t)$ is a form of

\[ \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}} \]
for \( S[f(t)] = G(u) \). Implies,
\[
G'(u) = \frac{G(u) - y(0)}{u}
\]
and
\[
G''(u) = \frac{S[f'(t)] - y'(0)}{u} = \frac{G(u) - y(0)}{u^2} - \frac{y'(0)}{u}
\]
are hold. Using the equation (*),
\[
S(ty') = 2uG'(u) - uy'(0) = 2G(u) - 2y(0) - uy'(0).
\]

(8) \( S(ty'') = 2uG''(u) - uy''(0) = \frac{2}{u}\{G(u) - y(0)\} - 2y'(0) - uy''(0) \)
\[
= \frac{2}{u}G(u) - \frac{2}{u}y(0) - 2y'(0) - uy''(0).
\]

(9) \( S(t^2y'') = u \frac{d}{du} [u S(ty'')] \)
\[
= u \frac{d}{du} [2G(u) - 2y(0) - 2uy'(0) - u^2y''(0)]
\]
\[
= u [2G'(u) - 2y'(0) - 2uy''(0)]
\]
\[
= 2uG'(u) - 2uy'(0) - 2u^2y''(0)
\]
\[
= 2G(u) - 2y(0) - 2uy'(0) - 2u^2y''(0)
\]
for \( S[f(t)] = G(u) \).

**Corollary 2.2**. (The alternate form (1), (2) and (3) of theorem 1)

1. \( \mathcal{L}(ty') = -Y - s \cdot dY/ds \) for \( Y = \mathcal{L}(y) \).
2. \( \mathcal{L}(ty'') = -s^2 \cdot dy/ds - 2sy + y(0) \).
3. \( \mathcal{L}(t^2y'') = dY/ds \cdot (s^3Y - s^2y(0) + 4s) \).

Proof. (1) Clear. (2) It follows from (**).

(3) \( \mathcal{L}(t^2y'') = - \frac{d}{ds} (\mathcal{L}(ty'')) = - \frac{d}{ds} (-s^2 \frac{dy}{ds} - 2sy + y(0)) \)
\[
= \frac{d}{ds} (s^2 \frac{dy}{ds} + 2sy - y(0)) = 2s \frac{dY}{ds} + s^2 \frac{d^2Y}{ds^2} + 2Y + 2s \frac{dY}{ds}.
\]

Since
\[
\frac{d^2Y}{ds^2} = \frac{dY}{ds} (sY - y(0)),
\]
we have

\[ \mathcal{L}(t^2 y'') = (s^3 Y - s^2 y(0) + 4s) \frac{dY}{ds}. \]

With the idea of theorem 1, we would like to check the following examples.

**Example 2.3**. Let us consider a Sturm-Liouville equation \( y'' + \lambda y = 0, \)
\( y(0) = a, \ y'(0) = b \) for \( \lambda \) is a given number.

**Solution** Taking the Laplace transform on both sides, we obtain

\[ s^2 Y - sy(0) - y'(0) + \lambda Y = 0 \]

for \( Y = \mathcal{L}(y) \). Organizing the equality, we have

\[ Y = \frac{sa + b}{s^2 + \lambda}. \]

Taking the inverse Laplace transform, we have the solution

\[ y = a \cos \sqrt{\lambda} t + \frac{b}{\sqrt{\lambda}} \sin \sqrt{\lambda} t. \]

**Example 2.4**. Let us consider Bessel’s equation

\[ y'' + \nu^2 y = 0, \ y(0) = 0, \ y'(0) = 1, \]

where \( \nu \) is a given number.

**Solution**. We can easily get the general solution

\[ y(t) = \frac{1}{\nu} \cdot \sin \nu t, \]  \hspace{1cm} (***)

where \( \nu = 1, 2, \cdots \). Next, let us take the Laplace transform on both sides of given equation. Then we have

\[ s^2 Y - sy(0) - y'(0) + \nu^2 Y = 0 \]

for \( Y = \mathcal{L}(y) = F(s) \). Organizing the equation, we have the solution

\[ y = \frac{\sin \nu t}{\nu} \]

for \( \nu \) is a given number[3]. Surely, this is the same result with (***).
Example 2.5. Let us consider the Euler-Cauchy equation \( t^2 y'' + aty' + by = 0 \).

**Solution.** 1) Applying to the original idea of Elzaki,

\[
E(t^2 y'') = u^4 \frac{d^2}{du^2} \left[ T(u) - u^2 f(0) - uf'(0) \right] \\
= u^4 \frac{d}{du} \left[ \frac{T'(u)u - 2T(u)}{u^3} - f'(0) \right] \\
= T''(u)u^2 - 4uT'(u) + 6T(u),
\]

\[
E(aty') = au^2 \frac{d}{du} \left[ T(u) - uf(0) \right] - au[T(u) - uf(0)] \\
= au^2 \left[ \frac{T'(u)u - T(u)}{u^2} - f(0) \right] \\
= auT'(u) - 2aT(u)
\]

and \( E[by] = bT(u) \). Arranging the equality, we get

\[
u^2 T''(u) + (a - 4)uT'(u) + (b - 2a + 6)T(u) = 0
\]

for \( E(f(t)) = T(u) \). Thus we have

\[
u^2[T(u) - u^2 f(0) - uf'(0)] + (a - 4)u[T(u) - uf(0)] \\
+ (b - 2a + 6)T(u) = 0.
\]

Collecting the \( T(u) \)-terms, we get

\[(b - a + 3)T(u) - (a - 3)f(0)u^2 - f'(0)u^3 = 0.\]

Since \( E(1) = u^2 \) and \( E(t) = u^3 \), we obtain

\[y = T^{-1}(u) = \frac{(a - 3)f(0) + tf'(0)}{b - a + 3} \quad [9].\]

2) Applying to the theorem 1, we simply get the equality

\[3T(u) + 3u^2 y(0) - u^3 y'(0)] - a[T(u) + u^2 y(0)] + bT(u) = 0,
\]

for \( E(f(t)) = T(u) \). Arranging the equality, we get

\[(b - a + 3)T(u) + (3 - a)u^2 y(0) - u^3 y'(0) = 0.\]

Thus, we have the solution

\[y = T^{-1}(u) = \frac{(a - 3)y(0) + ty'(0)}{b - a + 3}.\]

This shows that the above proposed theorem 1 is very practical for the application of integral transforms.
References


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