Study of Classes of Operators where the Distance of the Identity Operator and the Derivation Range is Maximal or Minimal

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Abstract

The main objective of this paper is to give more informations about classes of operators for which the distance of the identity operator and the derivation range is minimal (noted $JA(\mathcal{H})$), or maximal (noted $F(\mathcal{H})$).

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Introduction

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. The inner derivation induced by $A \in \mathcal{B}(\mathcal{H})$ is the map defined by: $\delta_A : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) / \delta_A(X) = AX - XA$. It is known that the identity operator $I$ is not a commutator i.e. For all $A \in \mathcal{B}(\mathcal{H})$; $I \not\in R(\delta_A)$, where $R(\delta_A)$ is the range of the derivation $\delta_A$.

Nevertheless J.H Anderson [1] proved the existence of an operator $B \in \mathcal{B}(\mathcal{H})$ such that: $I \in R(\delta_B)$, where $R(\delta_B)$ is the closure of $R(\delta_B)$ in the norm topology. This permitted him to define a new class of operators noted:

$$JA(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : I \in R(\delta_A)\} = \{A \in \mathcal{B}(\mathcal{H}) ; \exists (X_n) \in \mathcal{B}(\mathcal{H}) : AX_n - X_nA \to I\}.$$

Which is the class of operators where the distance of the identity operator and the derivation range is minimal.
J. P. Williams [21] introduced a class of operators, noted by:
\[ \mathcal{F}(\mathcal{H}) = \{ A \in \mathcal{B}(\mathcal{H}) : \| A X - X A - I \| \geq 1, \forall X \in \mathcal{B}(\mathcal{H}) \}. \]

Which is the class of operators where the distance between the identity operator and the derivation range is maximal, Williams proved that \( \mathcal{F}(\mathcal{H}) \) contain the normal, hyponormal, dominant and operators satisfies the quadratic equation. It is clear that \( \mathcal{F}(\mathcal{H}) \cap \mathcal{J}(\mathcal{H}) = \emptyset \), and \( \mathcal{F}(\mathcal{H}) \) is not the complementary of \( \mathcal{J}(\mathcal{H}) \), as there are elements of \( \mathcal{B}(\mathcal{H}) \) which are neither operators in \( \mathcal{J}(\mathcal{H}) \) nor operators in \( \mathcal{F}(\mathcal{H}) \). [10] proved the existence of such operators. In the nineties and beyond, several class of operators were introduced without study if they belong or not to the classes \( \mathcal{J}(\mathcal{H}) \) and \( \mathcal{F}(\mathcal{H}) \).

**Preliminary Notes**

In this paper we are aiming to investigate these classes and for this purpose we survey the following definitions.

An operator \( A \in \mathcal{B}(\mathcal{H}) \) is said to be normal if \( (A^*A) - (AA^*) = 0 \), hyponormal if \( (A^*A) - (AA^*) \geq 0 \), \( p \)-hyponormal if \( (A^*A)^p - (AA^*)^p \geq 0 \); for \( 0 < p \leq 1 \), paranormal if \( ||Ax||^2 \leq ||A^2x|| \); for all unit vector \( x \in \mathcal{H} \), and normaloid if \( ||A|| = r(A) \), where \( r(A) \) is the spectral radius of \( A \).

A \( A \in \mathcal{B}(\mathcal{H}) \) is said to be log-hyponormal if \( A \) is invertible and satisfies the following inequality: \( \log(A^*A) - \log(AA^*) \geq 0 \), class \( (A) \) if \( |A^2| - |A|^2 \geq 0 \) (where \( |A|^2 = A^*A \)), class \( \mathcal{A}(k) \) if \( \left| A^k |A|^{2k} A \right|^{1/k} \geq |A|^2 \), for \( k \in \mathbb{N} \), absolutely \( k \)-paranormal if \( \left| \|A\|^k Ax \right| \geq \|Ax\|^{k+1} \), \( \forall x \in \mathcal{H} \) and \( k \in \mathbb{N} \).

We say that \( A \) is transaloid if \( (\alpha - A) \) is normaloid for all \( \alpha \in \mathbb{C} \); moreover, if \( p(A) \) is normaloid for each polynomial \( p(\alpha) \), then we say that \( A \) is polynomial-normaloid.

Furthermore, if \( f(A) \) is normaloid for each rational function \( f \) with poles outside \( \sigma(A) \), then \( A \) is called a von Neumann operator.

The operator \( A \in \mathcal{B}(\mathcal{H}) \) belong to the class \( (W N) \) if \( (Re(A))^2 \leq |A|^2 \); where \( Re(A) \) is the real part of \( A \), also \( A \) is said to be a posinormal if there exist a positive operator \( P \in \mathcal{B}(\mathcal{H}) \) such that \( AA^* = A^*PA \).

An arbitrary operator \( A \in \mathcal{B}(\mathcal{H}) \) has a unique polar decomposition \( A = U |A| \), where \( U \) is a unitary operator, there is a very useful related operator associated with \( A \); \( \tilde{A} = |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}} \), called the Aluthge transformation of \( A \). \( A \) is said to be \( w \)-hyponormal if \( |\tilde{A}| \geq |A| \geq |\tilde{A}^*| \).

\( A \) is said to be a \( p \)-quasihyponormal operator if \( A^* \left[ (A^*A)^p - (AA^*)^p \right] A \geq 0 \); for \( 0 < p \leq 1 \). If \( p = 1 \), \( A \) is said to be a quasihyponormal operator. \( k \)-quasihyponormal operator if \( A^k [A^*A - AA^*] A^k \geq 0 \); for \( k \in \mathbb{N} \), and \( (p, k) \)-quasihyponormal operator if \( A^k [A^*A]^p - (AA^*)^p A^k \geq 0 \); for \( k \in \mathbb{N} \) and \( 0 < p \leq 1 \).

\( A \) is said to be a quasi-class \( (A) \) if \( A^* \left[ |A^2| - |A|^2 \right] A \geq 0 \), and \( k \)-quasi-class \( (A) \) if \( A^k \left[ |A^2| - |A|^2 \right] A^k \geq 0 \); for \( k \in \mathbb{N} \).

An operator \( A \) belongs to class \( A(s, t) \) for \( s > 0 \) and \( t > 0 \) if and only if:
\[ \left( |A^*|^t |A|^{2s} |A^*|^t \right)^{\frac{1}{2s+t}} \geq |A^*|^{2t} \]
and in the class \( w A(s, t) \) if and only if:
\[ |A^n|^{2s} \geq \left( |A^s|^{2t} |A^{s+t}| \right)^{\frac{s}{s+t}} \text{ and } \left( |A^s|^{2t} |A^{s+t}| \right)^{\frac{s}{s+t}} \geq |A^s|^{2t} \text{ for } s, t > 0. \]

An operator \( A \) belongs to class \( F(p, r, q) \) for \( p, r > 0 \) and \( q \geq 1 \) if and only if:
\[ \left( |A^r|^{2p} |A^{r+p}| \right)^{\frac{1}{q}} \geq |A^s|^{2(\frac{p+r}{q})}, \text{ and in the class } wF(p, r, q) \text{ if and only if} \]
\[ \left( |A^r|^{2p} |A^{r+p}| \right)^{\frac{1}{q}} \geq |A^s|^{2(p+r)}(1-\frac{q}{2}) \geq \left( |A^p| |A^s|^{2r} |A|^2 \right)^{1-\frac{1}{q}} \text{ for } p, r > 0 \text{ and } q \geq 1. \]

1. **Algebraic structure of \( \mathcal{J}A(\mathcal{H}) \) and \( \mathcal{F}(\mathcal{H}) \)**

Using the definition of \( \mathcal{F}(\mathcal{H}) \), it is easy to verify that \( I, 0 \in \mathcal{F}(\mathcal{H}). \) (So \( \not\in \mathcal{J}A(\mathcal{H}) \))

**Proposition 1.1** Let \( A \in \mathcal{B}(\mathcal{H}) \), and \( \alpha, \beta \in \mathbb{C}^* \) then
1. If \( A \in \mathcal{F}(\mathcal{H}) \) [or in \( \mathcal{J}A(\mathcal{H}) \)], then \( (\alpha A + \beta I) \in \mathcal{F}(\mathcal{H}) \).
2. If \( A \in \mathcal{F}(\mathcal{H}) \) [or in \( \mathcal{J}A(\mathcal{H}) \)], then \( A^* \in \mathcal{F}(\mathcal{H}) \).
3. If \( A \) is an invertible operator in \( \mathcal{F}(\mathcal{H}) \) [or in \( \mathcal{J}A(\mathcal{H}) \)], then \( A^{-1} \in \mathcal{F}(\mathcal{H}) \).

**Proof**
1. Clearly.
2. For \( X \in \mathcal{B}(\mathcal{H}) \), we have:
\[ \|A^*X - XA^* - I\| = \|(X^*A - AX^* - I)^*\| = \|X^*A - AX^* - I\| \text{ for } A \in \mathcal{F}(\mathcal{H}) \]
\[ \|A^*X - XA^* - I\| = \|A(-X^*) - A(-X^*) - I\| \geq 1. \]
So \( \|A^*X - XA^* - I\| \geq 1. \) We use the same argument for \( A \in \mathcal{J}A(\mathcal{H}) \).
3. If \( A \in \mathcal{F}(\mathcal{H}) \), for any \( X \in \mathcal{B}(\mathcal{H}) \), we have \( \|AX - XA - I\| \geq 1 \), thus if \( X = -A^{-1}YA^{-1} \) for any \( Y \in \mathcal{B}(\mathcal{H}) \):
\[ \|A^{-1}X - XA^{-1} - I\| \geq 1. \]
We use the same argument for \( A \in \mathcal{J}A(\mathcal{H}) \).

**Proposition 1.2** Let \( A, B \in \mathcal{B}(\mathcal{H}) \); then:
1. If \( A, B \in \mathcal{F}(\mathcal{H}) \), then \( AB \in \mathcal{F}(\mathcal{H}) \) and \( (A + B) \in \mathcal{F}(\mathcal{H}) \).
2. There exist \( A, B \in \mathcal{J}A(\mathcal{H}) \) such that \( AB \notin \mathcal{J}A(\mathcal{H}) \) and \( (A + B) \notin \mathcal{J}A(\mathcal{H}) \).

**Proof**
1. Let \( A, B \in \mathcal{F}(\mathcal{H}) \) and suppose that \( AB \notin \mathcal{F}(\mathcal{H}) \). using proposition 1.1 if \( A \in \mathcal{F}(\mathcal{H}) \); then \( A^{-1} \in \mathcal{F}(\mathcal{H}) \), so \( AA^{-1} = I \notin \mathcal{F}(\mathcal{H}) \); which is contradiction.
Suppose that \( A + B \notin \mathcal{F}(\mathcal{H}) \); if \( B = -A + \theta I \), \( \forall \theta \in \mathbb{C} \), then \( A + B = \theta I \notin \mathcal{F}(\mathcal{H}) \) which is contradiction.
2. We use the same argument for \( \mathcal{J}A(\mathcal{H}) \).
Unfortunately \( \mathcal{J}A(\mathcal{H}) \) has no significant algebraic structure, but \( \mathcal{F}(\mathcal{H}) \) is a field for operations sum and product as it is proved previously.

2. **Operators in \( \mathcal{J}A(\mathcal{H}) \)**

J. A. Anderson gave a necessary and sufficient conditions for \( A \in \mathcal{J}A(\mathcal{H}) \), Halmos and Yong Ho gave necessary or sufficient condition for \( A \in \mathcal{J}A(\mathcal{H}) \). Here we give a necessary and sufficient condition for \( A \in \mathcal{J}A(\mathcal{H}) \) and give the form of operators in the class \( \mathcal{J}A(\mathcal{H}) \).
Theorem 2.1 [15] Let \( A \in B(\mathcal{H}) \), and \( f \) an analytic function on an open set containing the spectrum of \( A \) to \( B(\mathcal{H}) \), such that \( f' \) does not vanish on \( \sigma(A) \). Then \( A \in \mathcal{J}\mathcal{A}(\mathcal{H}) \) if and only if \( f(A) \in \mathcal{J}\mathcal{A}(\mathcal{H}) \).

\[
i.e. I \in R(\delta A) \iff I \in R(\delta f(A)).
\]

Corollary 2.1 Let \( A \) be an invertible operator such that \( A \neq A^{-1} \), then \( A \in \mathcal{J}\mathcal{A}(\mathcal{H}) \) if and only if \( A^n \in \mathcal{J}\mathcal{A}(\mathcal{H}) \).

Proof Let \( f \) be a function defined by \( f(x) = x^n \), for \( A \in B(\mathcal{H}) \); \( f(A) \) is an analytic function, such that \( f' \) does not vanish on \( \sigma(A) \). Using the previous theorem; \( A \in \mathcal{J}\mathcal{A}(\mathcal{H}) \) if and only if \( A^n \in \mathcal{J}\mathcal{A}(\mathcal{H}) \).

Theorem 2.2 Let \( A \in B(\mathcal{H}) \) an operator of the form \( (A_{ii})_{1 \leq i \leq n} \), if for all \( i;1 \leq i \leq n \): \( A_{ii} \in \mathcal{J}\mathcal{A}(\mathcal{H}_i) \), then \( A \in \mathcal{J}\mathcal{A}(\bigoplus_{i=1}^{i=n} \mathcal{H}_i) \).

Proof Let \( A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_{nn} \end{pmatrix} \in B(\bigoplus_{i=1}^{i=n} \mathcal{H}_i) \), such that for all \( i;1 \leq i \leq n \): \( A_{ii} \in \mathcal{J}\mathcal{A}(\mathcal{H}_i) \) i.e. there exist a sequence \( (X_{ii}) \in B(\bigoplus_{i=1}^{i=n} \mathcal{H}_i) \): \( AX_{ii} - X_{ii}A \to I_{ii} \).

For \( X_n = \begin{pmatrix} X_{11} & 0 & \cdots & 0 \\ 0 & X_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_{nn} \end{pmatrix} \in B(\bigoplus_{i=1}^{i=n} \mathcal{H}_i) \); we have \( AX_n - X_nA = \begin{pmatrix} A_{11}X_{11} - X_{11}A_{11} & 0 & \cdots & 0 \\ 0 & A_{22}X_{22} - X_{22}A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_{nn}X_{nn} - X_{nn}A_{nn} \end{pmatrix} \)

for \( n \) tends to infinity we find: \( AX_n - X_nA \to \begin{pmatrix} I_{11} & 0 & \cdots & 0 \\ 0 & I_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_{nn} \end{pmatrix} \), the identity operator \( B(\bigoplus_{i=1}^{i=n} \mathcal{H}_i) \)

Proposition 2.1 [15] Let \( A,P \in B(\mathcal{H}) \), such that \( P^2 = P \), if \( I + P \in R(\delta A) \cap \{A\}' \), then \( A \in \mathcal{J}\mathcal{A}(\mathcal{H}) \).Where \( \{A\}' \) is the set of commutator of \( A \).
Proposition 2.2 [15] Let $A, P \in \mathcal{B}(\mathcal{H})$, such that $P^n \to 0$, if $I + P \in \overline{R(\delta_A) \cap \{A\}'}$, then $A \in \mathcal{J}A(\mathcal{H})$.

3. Operators in $\mathcal{F}(\mathcal{H})$

Let $A \in \mathcal{B}(\mathcal{H})$, we note by $\sigma_{ra}(A)$ the reduced approximate point spectrum of $A$, the set of scalars $\alpha$ for which there exist a sequence $(x_n) \in \mathcal{H}$, verifying: $\lim_{n \to +\infty} (A - \alpha I)x_n = 0$ and $\lim_{n \to +\infty} (A - \alpha I)^*x_n = 0$.

The approximate point spectrum of $A$, noted by $\sigma_a(A)$ is the set of scalars $\alpha$ for which there exist a sequence $(x_n) \in \mathcal{H}$, verifying: $\lim_{n \to +\infty} (A - \alpha I)x_n = 0$.

J.P.Williams [21] proved that $\overline{R_n} \subset \mathcal{F}(\mathcal{H})$; where $\overline{R_n}$ is the closure of the class of operators $A \in \mathcal{B}(\mathcal{H})$ which have a reducing subspace of dimension $n$; P.R. Halmos [10] proved that $\overline{R_1} = \{A \in \mathcal{B}(\mathcal{H}) : \sigma_{ra}(A) \neq \emptyset\}$.

Theorem 3.1 Let $A \in \mathcal{B}(\mathcal{H})$, if $A \in \overline{R_1}$; then $A \in \mathcal{F}(\mathcal{H})$.

Proof If $A \in \overline{R_1}$; then there exist a sequence $(x_n) \in \mathcal{H}$, such that $\lim_{n \to +\infty} (A - \alpha I)x_n = 0$ and $\lim_{n \to +\infty} (A - \alpha I)^*x_n = 0$. with $\|x_n\| = 1$, So if $B = A - \alpha I$ and $T = BX - XB$; where $X \in \mathcal{B}(\mathcal{H})$; then:

$$\|(A - \alpha I)X - (A - \alpha I)X - I\|^2 = \|BX - XB - I\|^2 \geq$$

$$\langle Tx_n - x_n; Tx_n - x_n \rangle \geq -2\Re \langle Tx_n; x_n \rangle + 1 \geq$$

$$-2\Re [\langle X x_n; B^* x_n \rangle - \langle B x_n; X^* x_n \rangle] + 1.$$ 

Now if $n$ tends to infinity we get the desired result.

Lemma 3.1 Let $A \in \mathcal{B}(\mathcal{H})$, if $A$ is normaloid operator; then $A \in \mathcal{F}(\mathcal{H})$.

Proof If $A$ is a normaloid operator, then $\|A\| = r(A)$ (the spectral radius of $A$); so there exist $\alpha \in \sigma(A)$ such that $\|A\| = |\alpha|$; so if: $\lim_{n \to +\infty} (A - \alpha I)x_n = 0$; then $\lim_{n \to +\infty} (A - \alpha I)^*x_n = 0$; so $\alpha \in \sigma_{ra}(A)$. Thus $A \in \mathcal{F}(\mathcal{H})$. As a direct consequence of Lemma 3.1 we obtain the following corollary.

Corollary 3.1 Let $A \in \mathcal{B}(\mathcal{H})$, if $A$ is a normal, hyponormal, p-hyponormal, paranormal, p-quasihyponormal, q-quasihyponormal, (p, q)-quasihyponormal, log-hyponormal, transaloid, polynomial-normaloid, von Neumann or w-hyponormal, class $A$, class $A(k)$ or absolutely k-paranormal, then $A \in \mathcal{F}(\mathcal{H})$.

Proof Since every given classes in the corollary is subset in the normaloid class.

Theorem 3.2 Let $A \in \mathcal{B}(\mathcal{H})$, if $A$ is a quasi class ($A$), k-quasi class ($A$), class $A(s, t)$, class $wA(s, t)$, class $F(p, r, q)$ or a class $wF(p, r, q)$ operator; then $A \in \mathcal{F}(\mathcal{H})$. 


Proof If $A$ is a quasi class $A$; we have [13] (resp $k$-quasi class $A$ [8]) : if 
$(A - \alpha I)x = 0$, then $(A - \alpha I)^* x = 0; \forall \alpha \in \mathbb{C}^*, \forall x \in \mathcal{H}$.

According to definitions, the following inclusions are verified:

$wA(s, t) \subset A(s, t)$ and $wF(p, r, q) \subset F(p, r, q)$, and for $A \in A(s, t)$, if $(A - \alpha I)x = 0$, then [19] $(A - \alpha I)^* x = 0; \forall \alpha \in \mathbb{C}^*, \forall x \in \mathcal{H}$ ...(1)

and the same for operators of class $F(p, r, q)$ [22]. It is obvious that if $\alpha$ satisfies 
(1), then $\alpha \in \sigma_{ra} (A)$, so $A \in \overline{M}_1$; which implies that $A \in \mathcal{F} (\mathcal{H})$.

Lemma 3.2 Let $A \in \mathcal{B} (\mathcal{H})$, if $A \in (WN)$; then $A \in \mathcal{F} (\mathcal{H})$.

Proof By [3] if $A \in (WN)$; then $A$ satisfies the property:

$$||(A - \alpha I)x|| \leq 3||(A - \alpha I)x|| \text{ for all } x \in \mathcal{H} \text{ and } \alpha \in \mathbb{C}; \text{ so for } \alpha \in \sigma_a (A) \text{ there exist a sequence } (x_n) \in \mathcal{H} \text{ such that: } \lim_{n \to +\infty} (A - \alpha I)x_n \to 0; \text{ then by the property we have: } \lim_{n \to +\infty} (A - \alpha I)^* x_n \to 0,$$

which means that $A \in \mathcal{F} (\mathcal{H})$.

Lemma 3.3 If $A \in \mathcal{B} (\mathcal{H})$ is a posinormal operator; then $A \in \mathcal{F} (\mathcal{H})$.

Proof. Let $A$ be a posinormal operator in $\mathcal{B} (\mathcal{H})$, then: $$||(A - \lambda I)^* x ||^2$$

$$= ((A - \lambda I)(A - \lambda I)^* x, x) \leq ((A - \lambda I)^* P(A - \lambda I)x, x) =$$

$$\langle \sqrt{P}(A - \lambda I)x, \sqrt{P}(A - \lambda I)x \rangle = \left\| \sqrt{P}(A - \lambda I)x \right\|^2 \leq \left\| \sqrt{P} \right\|^2 ||(A - \lambda I)x||^2$$

$$||(A - \lambda I)^* x || \leq \left\| \sqrt{P} \right\| ||(A - \lambda I)x||$$

now similar to Lemma 3.2 we have $A \in \mathcal{F} (\mathcal{H})$.

Lemma 3.4 [13] Let $A, B \in \mathcal{B} (\mathcal{H})$ such that $A^n = I$ and $B^n = I$, for $n \in \mathbb{N}$; 
then $\|AX - XB - T\| \geq \|T\|$; $\forall X \in \mathcal{B} (\mathcal{H})$, and all operator $T$ such that $AT = TB$.

Using lemma 3.4 for $A = B$ and $T = I$ we find:

Corollary 3.2 Let $A \in \mathcal{B} (\mathcal{H})$, such that $A^n = I$, for $n \in \mathbb{N}$; then: $\|AX - XA - I\| \geq 1; \forall X \in \mathcal{B} (\mathcal{H})$ i.e. $A \in \mathcal{F} (\mathcal{H})$.

Theorem 3.3 Let $A \in \mathcal{B} (\mathcal{H})$; such that $A = \left( \begin{array}{cc} S & 0 \\ 0 & T \end{array} \right) \in \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$; if 
$S \in \mathcal{F} (\mathcal{H}_1)$ or $T \in \mathcal{F} (\mathcal{H}_2)$; then $A \in \mathcal{F} (\mathcal{H})$.

Proof. Let $A = \left( \begin{array}{cc} S & 0 \\ 0 & T \end{array} \right)$, $X = \left( \begin{array}{cc} Z & Y \\ U & V \end{array} \right)$ and $I = \left( \begin{array}{cc} I_1 & 0 \\ 0 & I_2 \end{array} \right)$; then

$B = AX - XA - I = \left( \begin{array}{cc} SZ - ZS - I_1 & SY - YT \\ TU - US & TV - VT - I_2 \end{array} \right) = (B_{ij}); i, j = 1, 2.$

It is known that: $\|AX - XA - I\| \geq \max_i \|B_{ii}\|

then $\|AX - XA - I\| \geq \|SZ - ZS - I_1\|$, or $\|AX - XA - I\| \geq \|TV - VT - I_2\|.$

Since $S$ or $T \in \mathcal{F} (\mathcal{H}_i)$ i.e. $\|SZ - ZS - I_1\|$ or $\|TV - VT - I_2\| \geq 1$

Then $\|AX - XA - I\| \geq 1$; i.e. $A \in \mathcal{F} (\mathcal{H})$. 

Corollary 3.3 Let $A \in \mathcal{B}(\mathcal{H})$, be an operator of the matrix form $(A_{ij})_{1 \leq i \leq n}$ in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_n$, if there exist $i = 1, n$: $A_{ii} \in \mathcal{F}(\mathcal{H}_i)$; then $A \in \mathcal{F}(\mathcal{H})$.

Proof We use the same argument as the previous proof. In the following theorem we give the condition for which the tensor product of two operators is finite.

Theorem 3.4 Let $A \in \mathcal{F}(\mathcal{H}), W$ an invertible operator in $\mathcal{B}(\mathcal{H})$, then $A \otimes W \in \mathcal{F}(\mathcal{H} \otimes \mathcal{H})$.

Proof. Let $A \in \mathcal{F}(\mathcal{H})$, then $\|AX - XA - I\| \geq 1; \forall X \in \mathcal{B}(\mathcal{H})$

$$1 \leq \|AX - XA - I\| = \|(AX - XA - I) \otimes I\| = \|AX \otimes I - XA \otimes I - I \otimes I\| = \|AX \otimes WW^{-1} - XA \otimes W^{-1}W - I \otimes I\|$$

$$= \|(A \otimes W) (X \otimes W^{-1}) - (X \otimes W^{-1}) (A \otimes W) - I \otimes I\|$$

for $Y = X \otimes W^{-1}$, we have $\|(A \otimes W) Y - Y (A \otimes W) - I \otimes I\| \geq 1$. This proves that $A \otimes W \in \mathcal{F}(\mathcal{H} \otimes \mathcal{H})$.

Lemma 3.5 Let $\mathcal{A}$ be a $C^*$-algebra, and let $a \in \mathcal{A}$ be an element of the class $\overline{\mathcal{A}}_1$; then $\|ax - xa - e\| \geq 1$.

Proof. By Berberian theorem; it is known that there exist a *-isomorphism isométrique $\varphi : A \rightarrow B(H)$ that preserves order; so $\|ax - xa - e\| = \|\varphi(ax - xa - e)\| = \|\varphi(a)x - \varphi(x)a - I\|$. If $a \in \mathcal{A}$ an element of the class $\overline{\mathcal{A}}_1$; then $\varphi(a) \in \mathcal{B}(\mathcal{H})$ verifie ; $1 \leq \|\varphi(a)x - \varphi(x)a - I\| = \|ax - xa - e\|$.

J. G. Stampfli [18] proves that the operators of the form hyponormal + compact or Toeplitz + compact are in $\mathcal{F}(\mathcal{H})$, the next theorem is a generalisation of Stampfli theorem.

Theorem 3.5 Let $A \in \mathcal{B}(\mathcal{H})$; if $A = \overline{\mathcal{A}}_1 + K$, then $A \in \mathcal{F}(\mathcal{H})$, where $K$ is a compact operator.

Proof. Let $\mathcal{K}(\mathcal{H})$ be the closed space of all compact operators, the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is a $C^*$-algebra, for $A = \overline{\mathcal{A}}_1 + K \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$; then there exist a Hilbert space $\mathcal{H}$ and a *-isomorphism isométrique $\varphi : \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, by lemma 3.5 $\varphi(A)$ verifie ;

$1 \leq \|\varphi(A) X - \varphi(X)A - I\|$, so $A \in \mathcal{F}(\mathcal{H})$.

References


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