On Some Identities of $k$-Jacobsthal-Lucas Numbers

H. Campos$^1$, P. Catarino$^1$, A. P. Aires$^2$, P. Vasco$^3$ and A. Borges$^3$

Universidade de Trás-os-Montes e Alto Douro, UTAD, http://www.utad.pt
Quinta de Prados, 5000-801 Vila Real, Portugal
Department of Mathematics, School of Science and Technology

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Abstract

In this paper we present the sequence of the $k$-Jacobsthal-Lucas numbers that generalizes the Jacobsthal-Lucas sequence introduced by Horadam in 1988. For this new sequence we establish an explicit formula for the term of order $n$, the well-known Binet’s formula, Catalan’s and d’Ocagne’s Identities and a generating function.

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1 Introduction

Several recurrence sequences of positive integers have been object of study for many researchers. Examples of these are the Fibonacci, Lucas, Pell, Pell-Lucas, Modified Pell, Jacobhstal, Jacobsthal-Lucas sequences among others(see [8], [10], [12], [13]). About them there is a vast literature studying several properties, ones involving the well-known Binet’s formula, Catalan’s, Cassini’s and d’Ocagne’s identities and there is also a vast literature dedicated to the study of other properties involving each sequence (see [7] and [14]).

More recently, some of these sequences were generalized for any positive real number $k$: the study of the $k$-Fibonacci sequence, the $k$-Lucas sequence, the $k$-Pell sequence, the $k$-Pell-Lucas sequence, the Modified $k$-Pell sequence and the $k$-Jacobhstal sequence appeared (see [1], [11], [2], [4], [5], [6] and [3]).
In this paper we generalize the sequence of Jacobsthal-Lucas numbers and study by introducing the sequence of the $k$-Jacobsthal-Lucas numbers. We give an explicit formula for the term of order $n$ of this sequence, the well-know Binet’s formula, Catalan’s and d’Ocagne’s Identities and a generating function for this recurrence sequence.

2 Identities

Let us define the sequence of the $k$-Jacobsthal-Lucas numbers $\{j_{k,n}\}_{n \in \mathbb{N}}$ as follows:

$$j_{k,n+1} = kj_{k,n} + 2j_{k,n-1}$$

(1)

where the initial conditions are:

$$\begin{cases}
  j_{k,0} = 2 \\
  j_{k,1} = k
\end{cases}$$

(2)

for any positive real number $k$. If $k = 1$ we get the sequence of Jacobsthal-Lucas numbers defined by Horadam in [9]. The characteristic equation associated to the recurrence relation (1) is

$$x^2 = kx + 2$$

(3)

with roots $r_1$ and $r_2$ given by $r_1 = \frac{k+\sqrt{k^2+8}}{2}$ and $r_2 = \frac{k-\sqrt{k^2+8}}{2}$.

Note that $r_1r_2 = -2$; $r_1 + r_2 = k$ and $r_1 - r_2 = \sqrt{k^2 + 8}$. Associated to (1) the term of order $n$ of the $k$-Jacobsthal-Lucas sequence, can be written by the following identity $j_{k,n} = c_1r_1^n + c_2r_2^n$ for some constants $c_1, c_2$.

Solving the system of two linear equations corresponding to the initial conditions (2),

$$\begin{cases}
  2 = c_1 + c_2 \\
  k = c_1r_1 + c_2r_2,
\end{cases}$$

(4)

we obtain $c_1 = c_2 = 1$. So, we get the next Proposition:

**Proposition 2.1 (Binet’s Formula):** The $n$th $k$-Jacobsthal-Lucas number $j_{k,n}$ is given by

$$j_{k,n} = r_1^n + r_2^n,$$

(5)

where $r_1$ and $r_2$ are the roots of the characteristic equation (3) and $r_1 > r_2$. 
Proof. We use induction on $n$. Taking into account the initial conditions (2), we note that the equation (5) is valid for $n = 0$ and $n = 1$. Now assume that (5) is true for $0 \leq s \leq n$, that is, $j_{k,s} = r_{1}^{s} + r_{2}^{s}$, for every $s \in \{0, \ldots, n\}$. Using (1) and taking into account that $r_{1}r_{2} = -2$ we have

\[
\begin{align*}
  j_{k,n+1} &= k j_{k,n} + 2 j_{k,n-1} \\
  &= k (r_{1}^{n} + r_{2}^{n}) + 2 (r_{1}^{n-1} + r_{2}^{n-1}) \\
  &= r_{1}^{n-1} (k r_{1} + 2) + r_{2}^{n-1} (k r_{2} + 2) \\
  &= r_{1}^{n-1} ((r_{1} + r_{2}) r_{1} + 2) + r_{2}^{n-1} ((r_{1} + r_{2}) r_{2} + 2) \\
  &= r_{1}^{n-1} (r_{1}^{2} + r_{1} r_{2} + 2) + r_{2}^{n-1} (r_{1} r_{2} + r_{2}^{2} + 2) \\
  &= r_{1}^{n+1} + r_{2}^{n+1}.
\end{align*}
\]

Consequently, the Binet’s Formula is true for any positive integer $n$. □

The use of the Binet’s Formula (5) and the fact that $r_{1}r_{2} = -2$ allows us to obtain Catalan’s Identity.

**Proposition 2.2 (Catalan’s Identity):**

\[
\begin{align*}
  j_{k,n-r} j_{k,n+r} - j_{k,n}^2 &= (-2)^{n-r} (j_{k,r}^2 - (-2)^{r+2}).
\end{align*}
\]

Proof. We have

\[
\begin{align*}
  j_{k,n-r} j_{k,n+r} - j_{k,n}^2 &= (r_{1}^{n-r} + r_{2}^{n-r}) (r_{1}^{n+r} + r_{2}^{n+r}) - (r_{1}^{n} + r_{2}^{n})^2 \\
  &= (-2)^{n} \left( \frac{r_{2}}{r_{1}} \right)^{r} + (-2)^{n} \left( \frac{r_{1}}{r_{2}} \right)^{r} - 2(-2)^{n} \\
  &= (-2)^{n} \left( \frac{r_{2}^{2r}+r_{1}^{2r}-2(r_{1}r_{2})^{r}}{(r_{1}r_{2})^{r}} \right) \\
  &= (-2)^{n} \left( \frac{r_{2}^{2r}+r_{1}^{2r}-2(r_{1}r_{2})^{r}}{(r_{1}r_{2})^{r}} \right) \\
  &= (-2)^{n-r} (r_{2}^{2r} + r_{1}^{2r} - 2(r_{1}r_{2})^{r}) \\
  &= (-2)^{n-r} \left( r_{1}^{2} + r_{2}^{2} - 4(r_{1}r_{2})^{r} \right) \\
  &= (-2)^{n-r} \left( j_{k,r}^{2} - 4(-2)^{r} \right),
\end{align*}
\]

as required. □

Substituting $r = 1$ in Catalan’s Identity (6), yields

\[
\begin{align*}
  j_{k,n-1} j_{k,n+1} - j_{k,n}^2 &= (-2)^{n-1} \left( j_{k,1}^{2} - 4(-2) \right)
\end{align*}
\]

and using the initial condition $j_{k,1} = k$, we obtain the Cassini’s identity for $k$-Jacobsthal-Lucas sequence.
Proposition 2.3 (Cassini’s Identity):
\[ j_{k,n-1}j_{k,n+1} - j_{k,n}^2 = (-2)^{n-1}(k^2 + 8). \]  
(7)

The d’Ocagne’s identity can also be obtained from the Binet’s Formula (5) and the fact that \( r_1r_2 = -2 \) and \( m > n \).

Proposition 2.4 (d’Ocagne’s Identity): For \( m > n \),
\[ j_{k,m}j_{k,n+1} - j_{k,m+1}j_{k,n} = (-2)^{n\sqrt{k^2 + 8}} \left( j_{k,m-n} - 2^{n-m+1}(k + \sqrt{k^2 + 8})^{m-n} \right). \]

Proof. For \( m > n \), we have
\[
\begin{align*}
j_{k,m}j_{k,n+1} - j_{k,m+1}j_{k,n} &= \left( r_1^m + r_2^m \right) (r_1^{n+1} + r_2^{n+1}) - \left( r_1^{m+1} + r_2^{m+1} \right) (r_1^n + r_2^n) \\
&= (-2)^n \left( r_1^{m-n+r_2} + r_1^{r_2-m-n} - r_1^{m-n}r_1 - r_2^{m-n}r_2 \right) \\
&= (-2)^n \left( r_1^{m-n} (r_2 - r_1) + r_2^{m-n} (r_1 - r_2) \right) \\
&= (-2)^n \sqrt{k^2 + 8} \left( r_1^{m-n} + r_2^{m-n} - 2r_1^{m-n} \right) \\
&= (-2)^n \sqrt{k^2 + 8} \left( j_{k,m-n} - 2^{m-n} \frac{k + \sqrt{k^2 + 8}}{2} \right) \\
&= (-2)^n \sqrt{k^2 + 8} \left( j_{k,m-n} - 2^{n-m+1} \left( k + \sqrt{k^2 + 8} \right)^{m-n} \right).
\end{align*}
\]
as required. \( \square \)

The limit property stated in the following Proposition is also deduced using Binet’s Formula (5).

Proposition 2.5 For \( m > n \),
\[ \lim_{n \to \infty} \frac{j_{k,n}}{j_{k,n-1}} = r_1. \]  
(8)

Proof. We have
\[
\lim_{n \to \infty} \frac{j_{k,n}}{j_{k,n-1}} = \lim_{n \to \infty} \frac{r_1^n + r_2^n}{r_1^{n-1} + r_2^{n-1}}.
\]
Since \( \left| \frac{r_2}{r_1} \right| < 1 \), then \( \lim_{n \to \infty} \left( \frac{r_2}{r_1} \right)^n = 0 \) and therefore
\[
\lim_{n \to \infty} \frac{j_{k,n}}{j_{k,n-1}} = \lim_{n \to \infty} \frac{1 + \left( \frac{r_2}{r_1} \right)^n}{1 + \left( \frac{r_2}{r_1} \right)^n} \frac{1}{r_1} = \frac{1}{r_1},
\]
and the result follows. \( \square \)
3 Generating Function

In the next Proposition we present a generating function for the sequence of the \( k \)-Jacobsthal-Lucas numbers.

**Proposition 3.1 (Generating function of the \( k \)-Jacobsthal-Lucas numbers)**

\[
\hat{j}_k(x) = \frac{2 - kx}{1 - kx - 2x^2}
\]

**Proof.** Let us suppose that the \( k \)-Jacobsthal-Lucas numbers are the coefficients of a power series centered at the origin, that is convergent in \( \left[-\frac{1}{r_1}, \frac{1}{r_1}\right] \), taking in account the Proposition (2.5). To the sum of this power series, \( \hat{j}_k(x) \), we call generating function of the \( k \)-Jacobsthal-Lucas numbers. So we have

\[
\hat{j}_k(x) = j_{k,0} + j_{k,1}x + j_{k,2}x^2 + \cdots + j_{k,n}x^n + \cdots
\]

and then,

\[
\begin{align*}
&kx\hat{j}_k(x) = k(j_{k,0}x + j_{k,1}x^2 + j_{k,2}x^3 + \cdots + j_{k,n}x^{n+1} + \cdots) \\
&2x^2\hat{j}_k(x) = 2j_{k,0}x^2 + 2j_{k,1}x^3 + 2j_{k,2}x^4 + \cdots + 2j_{k,n}x^{n+2} + \cdots
\end{align*}
\]

Since (1) e (2) we obtain

\[
\hat{j}_k(x) - kx\hat{j}_k(x) - 2x^2\hat{j}_k(x) = 2 - kx
\]

and then we conclude that

\[
\hat{j}_k(x) = \frac{2 - kx}{1 - kx - 2x^2}
\]

\(\Box\)

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**References**


1 Member of CM-UTAD and Collaborator of CIDTFF, Portuguese Research Centers
2 Member of CIDTFF and Collaborator of CM-UTAD, Portuguese Research Centers
3 Member of CM-UTAD, Portuguese Research Center

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