Fixed Points for Generalized Condensing Maps in Abstract Convex Uniform Spaces

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Abstract

In this paper, we present some fixed point theorems for generalized condensing multimaps on KKM uniform spaces. We also obtain some new fixed point results for generalized condensing multimaps in \( \mathcal{KC} \) class (or KKM class) in the setting of abstract convex uniform spaces.

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1 Introduction and Preliminaries

The concept of condensing multimaps on locally convex topological vector spaces in Himmelberg et al. [2] was extended to locally \( G \)-convex uniform spaces by Huang et al. [4]. Recently, S. Park [6, 7, 8, 9] introduced the new concept of abstract convex spaces as a far-reaching generalization of convex spaces, \( H \)-spaces, generalized convex (or \( G \)-convex) spaces and other abstract convex structures.
The aim of this paper is to present some fixed point theorems for generalized condensing multimaps on abstract convex uniform spaces. We investigate the fixed point problems for maps on KKM uniform spaces in Section 2. We also obtain some new fixed point results for generalized condensing multimaps in $\mathcal{KC}$ class (or KKM class) in the setting of abstract convex uniform spaces in Section 3. $\mathcal{KC}$ class is equivalent to $s$-KKM class with surjective function $s$. So we reformulated the results in $\mathcal{KC}$ class to those in $s$-KKM class.

A multimap (or simply, a map) $F : X \to Y$ is a function from a set $X$ into the power set of $Y$; that is, a function with the values $F(x) \subset Y$ for $x \in X$ and the fibers $F^{-1}(y) := \{ x \in X \mid y \in F(x) \}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup \{ F(x) \mid x \in A \}$. Throughout this paper, we assume that multimaps have nonempty values otherwise explicitly stated or obvious from the context. The closure operation and graph of $F$ are denoted by $\overline{\text{co}} F$ and $\text{Gr} F$, respectively.

Let $\langle X \rangle$ denote the set of all nonempty finite subsets of a set $X$.

The followings are due to Park [6].

An abstract convex space $(X; \Gamma)$ consists of a topological space $X$, and a multimap $\Gamma : \langle X \rangle \to X$ with nonempty values. It is denoted that $\Gamma_A := \Gamma(A)$ for $A \in \langle X \rangle$.

For any nonempty $A \subset X$, the $\Gamma$-convex hull of $A$ is denoted and defined by

$$\text{co}_\Gamma A := \bigcup \{ \Gamma_N \mid N \in \langle A \rangle \} \subset X.$$ 

A subset $S$ of $X$ is called a $\Gamma$-convex subset of $(X; \Gamma)$ if for any $N \in \langle S \rangle$, we have $\Gamma_N \subset S$.

If $X$ is compact, then $(X; \Gamma)$ is called a compact abstract convex space.

An abstract convex uniform space $(X; \Gamma; U)$ is an abstract convex space with a basis $U$ of a uniform structure of $X$. $A \subset X$ and $U \in U$, the set $U[A]$ is defined to be $\{ y \in X : (x, y) \in U \text{ for some } x \in A \}$ and if $x \in X$, $U[x] = U[\{ x \}]$.

An abstract convex uniform space $(X; \Gamma; U)$ is called an $L\Gamma$-space if $U[C]$ is $\Gamma$-convex for each $U \in U$ whenever $C \subset X$ is $\Gamma$-convex.

For an abstract convex uniform space $(X; \Gamma; U)$, a subset $S$ of $X$ is said to be of the Zima type or of the Zima-Hadžić type if for each $U \in U$ there exists a $V \in U$ such that for each $N \in \langle S \rangle$ and any $\Gamma$-convex subset $A$ of $S$, we have

$$A \cap V[z] \neq \emptyset \quad \forall z \in N \implies A \cap U[x] \neq \emptyset \quad \forall x \in \Gamma_N.$$

A generalized convex space or a $G$-convex space $(X; \Gamma)$ consists of a topological space $X$ such that for each $A \in \langle X \rangle$ with the cardinality $|A| = n + 1$, there exist a subset $\Gamma_A$ of $X$ and a continuous map $\phi_A : \Delta_n \to \Gamma_A$ such that
A multimap \( T \) of precompactness of \( X \). Note that a generalized convex space is also a KKM space.

For details on \( G \)-convex spaces, see [10, 11, 12].

\( G \)-convex spaces are abstract convex spaces and other examples of abstract convex spaces are given in Park [6, 9] and the references therein.

A \( G \)-convex uniform space \((X; \Gamma; U)\) is a \( G \)-convex space with a basis \( U \) of a uniform structure of \( X \).

A \( G \)-convex uniform space \((X; \Gamma; U)\) is said to be a \emph{locally \( G \)-convex uniform space} if the uniformity \( U \) has a base \( B \) consisting of open symmetric entourages such that for each \( U \in B \)

1. \( U[x] \) is \( \Gamma \)-convex for each \( x \in X \); and
2. \( U[C] \) is \( \Gamma \)-convex for each \( U \in U \) whenever \( C \subset X \) is \( \Gamma \)-convex.

For details, see [13]. Note that locally \( G \)-convex uniform spaces are \( L\Gamma \)-spaces.

Let \((X; \Gamma)\) be an abstract convex space and \( Z \) be a set. For a multimap \( F : X \to Z \), if a multimap \( G : X \to Z \) satisfies \( F([A]) \subset G(A) \) for all \( A \in \langle X \rangle \), then \( G \) is called a \emph{KKM map} with respect to \( F \). A \emph{KKM map} \( G : X \to Z \) is a KKM map with respect to the identity map \( 1_X \).

A multimap \( F : X \to Z \) is called a \( \mathcal{R} \)-map if, for a KKM map \( G : X \to Z \) with respect to \( F \), the family \( \{G(x)\}_{x \in X} \) has the finite intersection property.

We denote \( \mathcal{R}(X, Z) := \{F : X \to Z | F \text{ is a } \mathcal{R} \text{-map}\} \).

Similarly, a \( \mathcal{R}\mathcal{E} \)-map is defined for closed-valued maps \( G \) and a \( \mathcal{R}\mathcal{D} \)-map for open-valued maps \( G \).

For an abstract convex space \((X; \Gamma)\), the \emph{KKM principle} is the statement \( 1_X \in \mathcal{R}(X, X) \cap \mathcal{R}(X, X) \). An abstract convex space is called a \emph{KKM space} if it satisfies the KKM principle.

Known examples of KKM spaces are given in [7, 9] and the references therein. Note that a generalized convex space is also a KKM space.

A subset \( S \) of a uniform space \( X \) is said to be \emph{precompact} if, for any entourage \( V \), there is a \( N \in \langle X \rangle \) such that \( S \subset V[N] \).

Motivated by Huang et al. [4], we extend the concepts of the measure of precompactness and condensing maps on locally convex topological vector spaces in Himmelberg et al. [2] to abstract convex uniform spaces.

For a subset \( A \) of an abstract convex uniform space \((X; \Gamma; U)\), a \emph{measure of precompactness} of \( A \) is denoted and defined by

\[
\Psi(A) = \{V \in U : A \subset V[S] \text{ for some precompact subset } S \text{ of } X\}.
\]

Let \( S \) be a nonempty subset of an abstract convex uniform space \((X; \Gamma; U)\). A multimap \( T : S \to X \) is called \emph{condensing} provided that \( \Psi(A) \subset \Psi(T(A)) \)
for any subset $A$ that is not precompact. $T$ is called generalized condensing if $A \subset S$, $T(A) \subset A$ and $A \setminus \text{cor} T(A)$ is precompact, then $A$ is precompact. Note that every compact map is condensing and every condensing map is generalized condensing.

2 Fixed points on KKM uniform spaces

Proposition 2.1. Let $(X;\Gamma;U)$ be an abstract convex space.

(a) If $x \in \Gamma_{\{x\}}$ for each $x \in X$, then for each $A \subset X$, $A \subset \text{co}\Gamma A$.

(b) If every singleton of $X$ is $\Gamma$-convex, then $\{x\} = \Gamma_{\{x\}}$ for each $x \in X$.

(c) Any $L\Gamma$-space is of the Zima type.

(a) and (b) are clear. (c) is in Park [8].

Proposition 2.2. Let $(X;\Gamma;U)$ be a Hausdorff abstract convex uniform space of the Zima type. If $C$ is a $\Gamma$-convex subset of $X$, then its closure $\overline{C}$ is $\Gamma$-convex.

Proof. For any $N = \{z_1, \cdots, z_n\} \in \langle \overline{C} \rangle$ and $V \in U$, there is $\{a_1, \cdots, a_n\} \subset C$ such that $z_i \in V[a_i]$, that means $C \cap U[a_i] \neq \emptyset$ for all $U \in U$ and $x \in \Gamma_N$. Hence $\Gamma_N \subset \bigcap_{U \in U} U[C] = \overline{C}$. \hfill $\square$

Proposition 2.3. Let $(X;\Gamma;U)$ be an abstract convex uniform space such that $x \in \Gamma_{\{x\}}$ for each $x \in X$ and $T : X \rightarrow X$ be a multimap.

(a) For a nonempty subset $Q$ of $X$, there is a $\Gamma$-convex subset $K$ of $X$ such that $\text{cor} (T(K) \cup Q) = K$.

(b) If $X$ is a complete Hausdorff space and $T$ is a generalized condensing closed multimap, then there is a nonempty precompact $\Gamma$-convex subset $K$ of $X$ such that $T(K) \subset K$ and $T(x) \cap \overline{K} \neq \emptyset$ for any $x \in \overline{K}$.

Proof. (a) Let $\mathcal{F} = \{A \subset X : A$ is $\Gamma$-convex and $\text{cor} (T(A) \cup Q) \subset A\}$. Since $X \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. Define a partial order by inverse inclusion, that is, for $A, B \in \mathcal{F}$, $A \leq B \iff B \subset A$. Let $\mathcal{C}$ be any chain in $\mathcal{F}$. Put $M = \bigcap_{A \in \mathcal{C}} A$. Since each $A \in \mathcal{C}$ is $\Gamma$-convex, $M$ is $\Gamma$-convex. Moreover, for all $A \in \mathcal{C}$, $T(M) \cup Q \subset T(A) \cup Q$, hence $\text{cor} (T(M) \cup Q) \subset \text{cor} (T(A) \cup Q) \subset A$, and so $\text{cor} (T(M) \cup Q) \subset \bigcap_{A \in \mathcal{C}} A = M$. Thus $M \in \mathcal{F}$ and $M$ is an upper bound of $\mathcal{C}$. By Zorn’s lemma, $\mathcal{F}$ has a maximal element, say $K$. We claim that $\text{cor} (T(K) \cup Q) = K$. In fact, put $K_0 = \text{cor} (T(K) \cup Q)$. Then $K_0$ is $\Gamma$-convex and $K_0 \subset K$. Furthermore, since $\text{cor} (T(K_0) \cup Q) \subset \text{cor} (T(K) \cup Q) = K_0$, we
have $K_0 \in \mathcal{F}$. By the maximality of $K$, we conclude that $K = K_0$, that is, $\text{co}_T(T(K) \cup Q) = K$.

(b) Choose $x_0 \in X$ and let $\Omega = \bigcup_{i \leq 0} T^i(x_0)$, where $T^0(x_0) = \{x_0\}$. Then $T(\Omega) \subset \Omega$ and $\Omega \setminus \text{co}_T T(\Omega) \subset \{x_0\}$, so $\Omega \setminus \text{co}_T T(\Omega)$ is precompact. As $T$ is generalized condensing, $\Omega$ is precompact, and hence $\Omega$ is compact since it is a precompact and closed subset of the complete set $X$.

Define $G : \overline{\Omega} \to \overline{\Omega}$ by $G(x) = T(x) \cap \overline{\Omega}$ for each $x \in \overline{\Omega}$. Since $T$ is closed and $\overline{\Omega}$ is compact, $G(x) \neq \emptyset$ for all $x \in \overline{\Omega}$, so $G$ is well defined. Put

$$\mathcal{A} = \{A : A \text{ is a nonempty closed subset of } \overline{\Omega} \text{ and } G(A) \subset A\}.$$ 

Since $\overline{\Omega} \in \mathcal{A}$, $\mathcal{A} \neq \emptyset$. Define a partial order $\leq$ on $\mathcal{A}$ by $A \leq B \iff B \subset A$, for any $A, B \in \mathcal{A}$. Let $\mathcal{C}$ be any chain in $\mathcal{A}$ and put $M = \cap_{A \in \mathcal{C}} A$. Then $M$ is a nonempty closed subset of $\overline{\Omega}$ and $G(M) = G(\cap_{A \in \mathcal{C}} A) \subset \cap_{A \in \mathcal{C}} G(A) \subset \cap_{A \in \mathcal{C}} A = M$. Thus $M$ is an upper bound of $\mathcal{C}$, and so, by Zorn’s Lemma, $\mathcal{A}$ has a maximal element, say $Q$, then $G(Q) \subset Q$. Since $T$ is closed, so is $G$, which in conjunction with the compactness of $\overline{\Omega}$, shows that $G$ is upper semicontinuous. Therefore $G(Q)$ is compact. Putting $Y = G(Q)$ and noting that $G(Y) = G(G(Q)) \subset G(Q) = Y$, the maximality of $Q$ gives us that $Q = Y$. Thus $Q = G(Q) = T(Q) \cap \overline{\Omega} \subset T(Q)$. Let $K = \text{co}_T(T(K) \cup Q)$ in (a). $Q \subset T(Q)$ and $Q \subset K$ imply that $Q \subset T(Q) \subset T(K)$. Hence $K = \text{co}_T(T(K) \cup Q) = \text{co}_T T(K)$, so $K$ is precompact, since $T(K) \subset K$ and $T$ is generalized condensing. Obviously, $K$ is $\Gamma$-convex and $\overline{K}$ is compact.

If $x \in K$, then we have $T(x) \cap \overline{K} \supset T(x) \cap K = T(x) \neq \emptyset$.

If $x \in \overline{K} \setminus K$, choose a net $\{x_\alpha\}$ in $K$ such that $x_\alpha \to x$. For any $\alpha$, choose $y_\alpha \in T(x_\alpha)$. Since $\{y_\alpha\} \subset T(K) \subset \overline{K}$ and $\overline{K}$ is compact, $\{y_\alpha\}$ has a subnet $\{y_{\alpha_j}\}$ such that $y_{\alpha_j} \to y$ for some $y \in \overline{K}$. Hence the closedness of $T$ implies that $y \in T(x)$. This shows that $y \in T(x) \cap \overline{K}$, completing the proof.

The proof of Proposition 2.3 is proceeded in the same fashion as Lemmas 3.4 and 3.5 in [4], except that the $\Gamma$-convex hull in a $G$-convex space is replaced by the $\Gamma$-convex hull in an abstract convex space.

**Proposition 2.4.** Let $(X; \Gamma; U)$ be a Hausdorff KKM uniform space and $T : X \to X$ be a compact upper semicontinuous map with closed $\Gamma$-convex values. If $T(X)$ is of the Zima type, then $T$ has a fixed point.

Proposition 2.4 is in [8].

**Theorem 2.5.** Let $(X; \Gamma; U)$ be a complete Hausdorff KKM uniform space of the Zima type with $x \in \Gamma\{x\}$ for each $x \in X$. And let $T : X \to X$ be a generalized condensing closed multimap with $\Gamma$-convex values. Then $T$ has a fixed point.
Proof. By Proposition 2.3, there exists a precompact \( \Gamma \)-convex subset \( K \) of \( X \) such that \( T(K) \subseteq K \) and \( T(x) \cap \overline{K} \neq \emptyset \) for any \( x \in \overline{K} \). Define \( F : \overline{K} \to \overline{K} \) by \( F(x) = T(x) \cap \overline{K} \) for \( x \in \overline{K} \). Since \( \overline{K} \) is \( \Gamma \)-convex by Proposition 2.2 and \( T \) has \( \Gamma \)-convex values, so is \( F \). Moreover \( F \) is compact and closed by the closedness of \( T \). So \( F \) is upper semicontinuous. By Proposition 2.4, \( F \) has a fixed point \( x_0 \) which is also a fixed point of \( T \).

Corollary 2.6. Let \((X; \Gamma; U)\) be a complete Hausdorff KKM LG-space such that every singleton is \( \Gamma \)-convex. And let \( T : X \to X \) be a generalized condensing closed multimap with \( \Gamma \)-convex values. Then \( T \) has a fixed point.

Corollary 2.6 generalizes and relaxes the conditions of Theorem 3.7 in [4].

3 Fixed point theorems for \( \mathcal{RC} \) maps

For a given abstract convex space \((X; \Gamma)\) and a topological space \( Y \), a map \( H : Y \to X \) is called a \( \Phi \)-map if there exists a map \( G : Y \to X \) such that

(i) for each \( y \in Y \), \( \text{co}_\Gamma G(y) \subseteq H(y) \); and

(ii) \( Y = \bigcup \{ \text{Int} G^{-1}(x) \ | \ x \in X \} \).

In \((X; \Gamma; U)\), a subset \( S \) of \( X \) is called a \( \Phi \)-set if, for any entourage \( U \in U \), there exists a \( \Phi \)-map \( H : S \to X \) such that \( \text{Gr} H \subseteq U \). If \( X \) itself is a \( \Phi \)-set, then it is called a \( \Phi \)-space.

The following Lemma is in [6];

Lemma 3.1. Let \((X; \Gamma)\) be an abstract convex space, \( C \) be a \( \Gamma \)-convex subset of \( X \) and \( Z \) be a set. If \( T \in \mathcal{R}(X, Z) \), then \( T|_C \in \mathcal{R}(C, Z) \).

The following Propositions are in [8];

Proposition 3.2. Let \((X; \Gamma; U)\) be an abstract convex uniform space such that every singleton is \( \Gamma \)-convex. Then any subset \( S \) of the Zima type in \( X \) is a \( \Phi \)-set.

Proposition 3.3. Let \((X; \Gamma; U)\) be a Hausdorff abstract convex uniform space, and \( T \in \mathcal{RC}(X, X) \) be a compact closed map. If \( \overline{T(X)} \) is a \( \Phi \)-set, then \( T \) has a fixed point.

Theorem 3.4. Let \((X; \Gamma; U)\) be a complete Hausdorff abstract convex uniform space of the Zima type such that every singleton is \( \Gamma \)-convex. Let \( T \in \mathcal{RC}(X, X) \) be a generalized condensing closed map, satisfying that \( T(K) \subseteq T(K) \) for any precompact \( \Gamma \)-convex subset \( K \) of \( X \). Then \( T \) has a fixed point.

Proof. By Proposition 2.3, there exists a precompact \( \Gamma \)-convex subset \( K \) of \( X \) such that \( T(K) \subseteq K \), which in conjunction with \( T(K) \subseteq \overline{T(K)} \) shows that \( T(K) \subseteq \overline{K} \). Then \( T|_\overline{K} \) is compact closed. Since \( \overline{K} \) is \( \Gamma \)-convex, \( T|_\overline{K} \in \mathcal{RC}(\overline{K}, \overline{K}) \). By Propositions 3.2 and 3.3, \( T|_\overline{K} \) has a fixed point. \( \square \)
Theorem 3.5. Let \((X; \Gamma; \mathcal{U})\) be a complete Hausdorff abstract convex uniform space of the Zima type such that every singleton is \(\Gamma\)-convex. Let \(T \in \mathcal{K}(X, X)\) be a lower semicontinuous generalized condensing closed map. Then \(T\) has a fixed point.

Proof. Since \(T\) is lower semicontinuous, \(\overline{T(K)} \subset \overline{T(K)}\) for any subset \(K\) of \(X\), so the conclusion follows from Theorem 3.4. \(\square\)

Theorem 3.6. Let \((X; \Gamma; \mathcal{U})\) be a complete Hausdorff abstract convex uniform space of the Zima type such that every singleton is \(\Gamma\)-convex. Let \(T : X \rightrightarrows X\) be a generalized condensing closed map satisfying the following; for any precompact \(\Gamma\)-convex subset \(A\) of \(X\), if \(F(x) = T(x) \cap \overline{A} \neq \emptyset\) for any \(x \in \overline{A}\), then \(F \in \mathcal{K}(\overline{A}, \overline{A})\). Then \(T\) has a fixed point.

Proof. Since \(T\) is a generalized condensing closed map, there is a precompact \(\Gamma\)-convex subset \(K\) of \(X\) such that \(T(x) \cap \overline{K} \neq \emptyset\) for any \(x \in \overline{K}\). Then \(F \in \mathcal{K}(\overline{K}, \overline{K})\) and \(F\) is compact closed, so \(F\) has a fixed point which is also a fixed point of \(T\). \(\square\)

Influenced by Chang et al. [1] we define the class of \(S\)-KKM maps: Let \(X\) be a nonempty set, \((Y; \Gamma)\) be an abstract convex space and \(Z\) be a topological space. If \(S : X \rightrightarrows Y\), \(T : Y \rightrightarrows Z\) and \(F : X \rightrightarrows Z\) are three multimaps satisfying

\[
T(co_S(A)) \subset F(A) \quad \text{for all } A \in \{X\},
\]

then \(F\) is called an \(S\)-KKM map with respect to \(T\). If for any \(S\)-KKM map \(F\) with respect to \(T\), the family \(\{F(x)\}_{x \in X}\) has the finite intersection property, then \(T\) is said to have the \(S\)-KKM property. The class \(S\)-KKM \((X, Y, Z)\) is defined to be the set \(\{T : Y \rightrightarrows Z \mid T\) has the \(S\)-KKM property\}. In the case where \(X = Y\) and \(S\) is the identity map \(1_X\), then \(S\)-KKM\((X, Y, Z) = \mathcal{K}(X, Z)\).

It was shown that \(\mathcal{K}(Y, Z) \subset S\)-KKM\((X, Y, Z)\) and was given an example showing that \(T \in S\)-KKM\((X, X, X)\) but \(T \notin \mathcal{K}(X, X)\) in [1]. Let \(X = Y = Z = [0, 1]\) and let \(S : X \rightarrow X\), \(T : X \rightrightarrows X\) be defined for \(x \in X\), by \(S(x) = \frac{x}{3}\) and

\[
T(x) = \begin{cases} 
\{0\} & x \in [0, \frac{1}{3}); \\
\{0, 1\} & x \in [\frac{1}{3}, \frac{2}{3}); \\
\{1\} & x \in [\frac{2}{3}, 1]. 
\end{cases}
\]

Even though \(T \notin \mathcal{K}(X, X)\), \(T \in \mathcal{K}(S(X), X) \cap S\)-KKM\((X, S(X), X)\). The following Proposition shows the relation between \(S\)-KKM maps and \(\mathcal{K}\)-maps more specifically:

Proposition 3.7. Let \(X\) be a nonempty set, \((Y; \Gamma)\) be an abstract convex space and \(Z\) be a topological space. For any surjective function \(s : X \rightarrow Y\), \(T \in \mathcal{K}(Y, Z)\) if and only if \(T \in s\)-KKM\((X, Y, Z)\).
Proof. It is enough to show that \( T \in s\text{-KKM}(X,Y,Z) \) implies \( T \in \mathcal{K}(Y,Z) \).
Suppose that \( T \in s\text{-KKM}(X,Y,Z) \), but \( T \notin \mathcal{K}(Y,Z) \). Then there is a closed valued map \( F : Y \to Z \) such that
\[
T(\text{co}_\Gamma A) \subset F(A)
\]
for all \( A \in \langle Y \rangle \) but for some \( \{y_1, \ldots, y_n\} \in \langle Y \rangle \), \( \bigcap_{i=1}^{n} F(y_i) = \emptyset \). Define \( G : X \to Z \) by \( G(x) = F(s(x)) \) for all \( x \in X \), then \( G \) is nonempty and closed values. By (1),
\[
G(N) = F(s(N)) \supset T(\text{co}_\Gamma s(N))
\]
for each \( N \in \langle X \rangle \). Since \( T \in s\text{-KKM}(X,Y,Z) \),
\[
\bigcap_{x \in N} G(x) = \bigcap_{x \in N} \text{co}G(x) \neq \emptyset
\]
for each \( N \in \langle X \rangle \). For each \( i = 1, \ldots, n \), choose \( x_i \in s^{-1}(y_i) \), then
\[
\emptyset \neq \bigcap_{i=1}^{n} G(x_i) = \bigcap_{i=1}^{n} F(s(x_i)) = \bigcap_{i=1}^{n} F(y_i),
\]
this is a contradiction. Therefore \( T \in \mathcal{K}(Y,Z) \). \( \square \)

Proposition 3.7 generalizes Proposition 2.4 in [5]. By Proposition 3.7, Theorems 3.4 - 3.6 are reformulated as follows;

**Theorem 3.8.** Let \( Z \) be a topological space, \( (X; \Gamma; \mathcal{U}) \) be a complete Hausdorff abstract convex uniform space of the Zima type such that every singleton is \( \Gamma \)-convex and \( s : Z \to X \) be a surjection. Let \( T \in s\text{-KKM}(Z,X,X) \) be a generalized condensing closed map, satisfying that \( T(K) \subset T(K) \) for any precompact \( \Gamma \)-convex subset \( K \) of \( X \). Then \( T \) has a fixed point.

**Theorem 3.9.** Let \( Z \) be a topological space, \( (X; \Gamma; \mathcal{U}) \) be a complete Hausdorff abstract convex uniform space of the Zima type such that every singleton is \( \Gamma \)-convex and \( s : Z \to X \) be a surjection. Let \( T \in s\text{-KKM}(Z,X,X) \) be a lower semicontinuous generalized condensing closed map. Then \( T \) has a fixed point.

**Theorem 3.10.** Let \( Z \) be a topological space, \( (X; \Gamma; \mathcal{U}) \) be a complete Hausdorff abstract convex uniform space of the Zima type such that every singleton is \( \Gamma \)-convex and \( s : Z \to X \) be a surjection. Let \( T : X \to X \) be a generalized condensing closed map satisfying the following; for any precompact \( \Gamma \)-convex subset \( A \) of \( X \), if \( F(x) = T(x) \cap \overline{A} \neq \emptyset \) for any \( x \in \overline{A} \), then \( F \in s\text{-KKM}(s^{-1}(A), \overline{A}, \overline{A}) \). Then \( T \) has a fixed point.

Theorems 3.8 - 3.10 generalize and delete some extra conditions in Theorems 2.7 - 2.9 in [3].

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