Unsteady Two-Dimensional Laminar Flow of a Viscous Fluid Between Two Parallel Porous Plates in Presence of a Transverse Magnetic Field

S. Ganesh

Department of Mathematics
Sathyabama University, Chennai – 600 119, Tamil Nadu, India

V. W. J. Anand

Department of Mathematics,
Sathyabama University, Chennai – 600 119, Tamil Nadu, India

Abstract

The unsteady magnetohydrodynamic flow between two parallel porous plates is a classical problem whose solution has many applications in Magnetohydrodynamic (MHD) power generators, cooling system, aerodynamics heating, polymer technology, petroleum industry, centrifugal separation of matter from fluid, purification of crude oil and fluid droplets and sprays.

Keywords: Unsteady laminar flow, magnetohydrodynamic flow, Suction Reynolds number, Hartmann number M, Pressure gradient

1 Introduction

Hassanien and Mansour (1990) discussed the unsteady magnetic flow through a porous medium between two infinite parallel plates. Bagchi (1996) studied the problem of Unsteady flow of viscoelastic Maxwell fluid with transient pressure gradient through a rectangular channel. Attia and Kotb (1996) studied the Steady,
fully developed MHD flow and heat transfer between two parallel plates with
temperature dependant viscosity. Aboul-Hassan and Attia (2002) discussed the
flow of a conducting viscoelastic fluid between two horizontal porous plates in the
presence of a transverse-magnetic field.

Attia (2004) has considered the Unsteady Hartmann flow with heat transfer of
a viscoelastic fluid considering the Hall effect. Hazeem Ali Attia (2005) studied
the unsteady laminar flow of an incompressible viscous fluid and heat transfer
between two parallel plates in the presence of a uniform suction and injection
considering variable properties. Ganesh and Krishnambal (2007) discussed the
Unsteady stokes flow of viscous fluid between two parallel porous plates. Ismail
and Ganesh studied the Unsteady Stokes Flow of Dusty fluid between two parallel
plates through porous medium.

The objective of this chapter is to analyze the unsteady magnetohydrodynamic
Stokes flow of viscous fluid between two parallel porous plates when the fluid is
being withdrawn through both the walls of the channel at the same rate. The
problem is reduced to a third order nonlinear differential equation which depends
on a Suction Reynolds number R and a Hartmann number M for which an exact
solution is obtained.

2 Mathematical Formulation

The unsteady laminar flow of an incompressible viscous fluid between two
parallel porous plates is considered in the presence of a transverse magnetic field
Ho applied perpendicular to the walls. The origin is taken at the centre of the
channel and let x and y be the coordinate axes parallel and perpendicular to the
channel walls. The length of the channel is assumed to be L and 2h is the distance
between the two plates. Let u and v be the velocity components in the x and y
directions respectively.

The equation of continuity is
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  \hspace{1cm} (1)

Equations of momentum are
\[ \rho \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \sigma_e B_0^2 u \]  \hspace{1cm} (2)
\[ \rho \frac{\partial v}{\partial t} = - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \sigma_e B_0^2 v \]  \hspace{1cm} (3)

where \( \sigma_e \) is the electrical conductivity and \( B_0 = \mu_e H_0 \), \( B_0 \) is the electromagnetic
induction, \( \mu_e \) being the magnetic permeability and \( H_0 \) is the transverse magnetic
field. The boundary conditions are \( u = 0 \) on \( y = h \) and \( y = -h \); \( v = v_0 e^{i\omega t} \) on \( y = h \)
and \( v = -v_0 e^{i\alpha t} \) on \( y = -h \) where \( v_0 \) is the velocity of suction at the walls of the channel.

Let \( \eta = \frac{y}{h} \), \( u = u(x,y) e^{i\alpha t} \), \( v = v(x,y) e^{i\alpha t} \), \( p = p(x,y) e^{i\alpha t} \) where \( \alpha \) is the frequency. The equations (1), (2) and (3) become

\[ \frac{\partial u}{\partial x} + \frac{1}{h} \frac{\partial v}{\partial \eta} = 0 \]  

(4)

\[ \rho \omega u = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{1}{h^2} \frac{\partial^2 u}{\partial \eta^2} \right) - \sigma_e B_0^2 u \]  

(5)

\[ \rho \omega v = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{1}{h^2} \frac{\partial^2 v}{\partial \eta^2} \right) - \sigma_e B_0^2 v \]  

(6)

Let \( \nu = \frac{\mu}{\rho} \) = Kinematic Viscosity, \( \rho \) the density of the fluid, \( \mu \) the coefficient of viscosity and \( p \) the pressure. The boundary conditions are converted into

\[ u(x,1) = 0, u(x,-1) = 0 \text{ and } v(x,1) = v_0, v(x,-1) = -v_0 \]  

(7)

Let \( \psi \) be the stream function such that \( u = \frac{1}{h} \frac{\partial \psi}{\partial \eta} \) and \( v = -\frac{\partial \psi}{\partial x} \)  

(8)

The equation of continuity can be satisfied by a stream function of the form

\[ \psi(x,\eta) = \left( hU(0) - v_0 x \right) f(\eta) \]  

(9)

where \( U(0) \) is the average entrance velocity at \( x = 0 \). From equation (9), the velocity components (8) given by

\[ u = \frac{1}{h} \left( hU(0) - v_0 x \right) f'(\eta) \text{ and } v = v_0 f(\eta) \]  

(10)

where the prime denotes the differentiation with respect to the dimensionless variable \( \eta = \frac{y}{h} \). Since the fluid is being withdrawn at constant rate from both the walls, \( v_0 \) is independent of \( x \). Using (10) in (5) and (6), the equations of momentum reduce to

\[ -\frac{1}{\rho} \frac{\partial p}{\partial x} = \left( U(0) - \frac{v_0 x}{h} \right) \left[ i \omega f'(\eta) - \frac{\nu}{h^2} f''''(\eta) + \frac{\sigma_e B_0^2}{\rho} f(\eta) \right] \]  

(11)

\[ -\frac{1}{h \rho} \frac{\partial p}{\partial \eta} = i \omega v_0 f(\eta) - \frac{\nu \cdot v_0 f''(\eta)}{h^2} + \frac{\sigma_e B_0^2}{\rho} v_0 f(\eta) \]  

(12)
Now differentiating (12) w.r.t. ‘x’, we get
\[
\frac{\partial^2 p}{\partial x^2 \eta} = 0 \tag{13}
\]

Differentiating (11) w.r.t. ‘\eta’, it follows that
\[
\frac{-1}{\rho} \frac{\partial^2 p}{\partial x \partial \eta} = \left( \frac{U(0)}{v} - \frac{v_x}{h} \right) \frac{d}{d\eta} \left( io f'(\eta) - \frac{v}{h^2} f''(\eta) + \frac{\sigma_z B_0^2}{\rho} f'(\eta) \right) \tag{14}
\]

From (13), equation (14) can be written as
\[
\frac{d}{d\eta} \left( io f'(\eta) - \frac{v}{h^2} f''(\eta) + \frac{\sigma_z B_0^2}{\rho} f'(\eta) \right) = 0 \text{ which is true for all } x \tag{15}
\]

Let \( R = \text{ Suction Reynolds number } = \frac{hv_0}{v} \) and \( M = \text{ Hartmann number } = B_0h \left( \frac{\sigma_z}{\rho \nu} \right)^{\frac{1}{2}} \)

Integrating (15) w.r.t. \( \eta \) and substituting the above expressions, it follows that
\[
\Rightarrow f''(\eta) - \left( io h^2 + \frac{\sigma_z B_0^2}{\rho \nu} h^2 \right) f'(\eta) = K \Rightarrow f''(\eta) - \alpha^2 h^2 f'(\eta) - a_i R f'(\eta) = K \tag{16}
\]

where \( R = \frac{hv_0}{v} \) and \( a_i = \frac{\sigma_z h B_0^2}{\rho \nu_0} \) and \( K \) is an arbitrary constant.

Boundary conditions are \( f(1) = 1, \ f(-1) = -1, \ f'(1) = 0 \) and \( f'(-1) = 0 \) \( \tag{17} \)

Hence the solution of the equations of motion and continuity is given by a non linear third order differential equation (16) subject to the boundary conditions (17). Equation (16) can be rewritten as
\[
f''(\eta) - \left( \alpha^2 h^2 + M^2 \right) f'(\eta) = K
\]

Where \( a_i R = M^2 \) and \( \alpha^2 = \frac{i \rho \omega}{\mu} \). i.e. \( D^3 - (\alpha^2 h^2 + M^2) D \) \( f(\eta) = K \)

\[
\therefore f(\eta) = A + B e^{\sqrt{\alpha^2 h^2 + M^2} \eta} + C e^{-\sqrt{\alpha^2 h^2 + M^2} \eta} - \frac{K \eta}{\alpha^2 h^2 + M^2} \tag{18}
\]

In order to obtain the values of the arbitrary constants, the boundary conditions \( f(1) = 1, \ f(-1) = -1, \ f'(1) = 0, \ f'(-1) = 0 \) are applied on \( f(\eta) \) and \( f'(\eta) \) and hence we get \( A = 0, \)

\[
B = \frac{1}{2 \sinh (\sqrt{\alpha^2 h^2 + M^2}) \left( 1 - \sqrt{\alpha^2 h^2 + M^2} \ coth \sqrt{\alpha^2 h^2 + M^2} \right)}
\]

\[
C = \frac{\left( \alpha^2 h^2 + M^2 \right)^{\frac{1}{2}} \ sinh \sqrt{\alpha^2 h^2 + M^2} \left( 1 - \sqrt{\alpha^2 h^2 + M^2} \ coth \sqrt{\alpha^2 h^2 + M^2} \right)}{2 \left( \alpha^2 h^2 + M^2 \right) \ sinh \sqrt{\alpha^2 h^2 + M^2} \left( 1 - \sqrt{\alpha^2 h^2 + M^2} \ coth \sqrt{\alpha^2 h^2 + M^2} \right)}
\]

\[
K = \frac{1}{1 - \sqrt{\alpha^2 h^2 + M^2} \ coth \sqrt{\alpha^2 h^2 + M^2}} \left( \frac{\sinh \sqrt{\alpha^2 h^2 + M^2} \eta}{\sinh \sqrt{\alpha^2 h^2 + M^2}} \right)
\]

\[
f(\eta) = \frac{1}{1 - \sqrt{\alpha^2 h^2 + M^2} \ coth \sqrt{\alpha^2 h^2 + M^2}} \left( \frac{\sinh \sqrt{\alpha^2 h^2 + M^2} \eta}{\sinh \sqrt{\alpha^2 h^2 + M^2}} \right)
\]

(19)
Hence the expressions for the velocity components are

\[ u = \frac{1}{h} [hU(0) - v_o x] f'(\eta) e^{i\omega t} \]

\[ = \left( U(0) - \frac{v_o x}{h} \right) \left( \frac{\sqrt{\alpha^2 h^2 + M^2}}{1 - \sqrt{\alpha^2 h^2 + M^2} \coth(\alpha^2 h^2 + M^2)} \right) \left( \frac{\cosh(\alpha^2 h^2 + M^2 \eta)}{\sinh(\alpha^2 h^2 + M^2)} - \frac{\cosh(\alpha^2 h^2 + M^2 \eta)}{\sinh(\alpha^2 h^2 + M^2)} \right) e^{i\omega t} \tag{20} \]

\[ v = v_o f(\eta) e^{i\omega t} \]

\[ = v_o \left( \frac{1}{1 - \sqrt{\alpha^2 h^2 + M^2} \coth(\alpha^2 h^2 + M^2)} \right) \left( \frac{\sinh(\alpha^2 h^2 + M^2 \eta)}{\sinh(\alpha^2 h^2 + M^2)} - \eta \frac{\sinh(\alpha^2 h^2 + M^2 \eta)}{\sinh(\alpha^2 h^2 + M^2)} \right) e^{i\omega t} \tag{21} \]

3 Pressure Distribution

From equation (11), it is seen that

\[ \frac{h^2}{\rho v} \frac{\partial p}{\partial x} = \left( U(0) - \frac{v_o x}{h} \right) \left( f''(\eta) - \frac{h^2}{\rho} \left( i \omega + \frac{\sigma B_s^2}{\rho} \right) f'(\eta) \right) \]

and since \( f''(\eta) = -\alpha^2 h^2 f'(\eta) - \omega R f'(\eta) = K \) (from 16)

i.e. \( f''(\eta) = -\frac{h^2}{\rho^2} \frac{\partial^2 p}{\partial x^2} - \frac{\sigma B_s^2 h v_o f(\eta)}{\rho} = K \)

\[ \therefore \frac{\partial^2 p}{\partial x^2} = K \frac{\rho v}{h^2} \left( U(0) - \frac{v_o x}{h} \right) = K \mu \left( U(0) - \frac{v_o x}{h} \right) \cdot \left( \frac{\mu}{\rho} \right) \tag{22} \]

Now, from equation (12) it is seen that

\[ \frac{\partial p}{\partial \eta} = f''(\eta) - i \omega v_o h \rho f(\eta) - \sigma B_s^2 h v_o f(\eta) \]

\[ \Rightarrow \frac{\partial p}{\partial \eta} = f''(\eta) - i \omega v_o h \rho f(\eta) - \sigma B_s^2 h v_o f(\eta) \]

Integrating (24), it is clearly seen that

\[ p(x, \eta) = K \mu \left( \left( U(0) - \frac{v_o x}{h} \right) \frac{\mu v_o}{h} \int_0^\eta f'(\eta) d\eta - \sigma B_s^2 h v_o \int_0^\eta f(\eta) d\eta \right) + K_1 \tag{25} \]

\[ \therefore \text{The pressure drop is given by} \]

\[ p(x, \eta) - p(0, 0) = K \mu \left( \left( U(0) - \frac{v_o x}{h} \right) \frac{\mu v_o}{h} \int_0^\eta f'(\eta) d\eta - \sigma B_s^2 h v_o \int_0^\eta f(\eta) d\eta \right) + K_1 \]

4 Discussion

The graphs of the axial velocity and radial velocity profiles have been drawn for different values of M. Figures 1 and 2 represents the axial velocity profiles at
different cross sections of the channel namely at $x = 0$ and $x = 4$ when the average entrance velocity is $u_0 = 0.5$ and $h = 1.0$. The magnitude of the axial velocity increases as $x$ increases from $x = 0$ to $x = 4$ for different values of $\omega t$, namely $\omega t = 0, \pi / 4, \pi / 2, 3\pi / 4$, and $\pi$ respectively.

The Figure 3 represents the axial velocity profiles of $u$ at $x=0, h=1.0$ when the inlet velocity is increased to $u_0 = 1.0$ from $u_0 = 0.5$. It is clearly seen that the magnitudes of the axial velocity $u$ are more when the values of $x$ are increased and also the magnitudes of the axial velocity $u$ are more when the inlet velocity is increased. When the Hartmann number $M$ is increased from $M = 1$ to $M = 2$ we see that the magnitude of the axial velocity profiles decrease as seen in Figure 4.

The Figures 5 and 6 represents the radial velocity profiles of $v$ at $v_0 = 0.5$, $h=1.0, \alpha=1.0$ and for different values of $M$. As $M$ increases from $M=1.0$ to $M=5.0$, it is seen that there is a marginal increase in the magnitude of the radial velocity. Here we see that the profiles are parabolic.
Unsteady two-dimensional laminar flow

Conclusion

In the above analysis a class of solutions of unsteady magneto hydrodynamics stokes flow of viscous fluid between two parallel porous plates is presented, in the presence of a transverse magnetic field when the fluid is being withdrawn through both the walls of the channel at the same rate. The above results reduce to the result of Ganesh & Krishnambal (2007) when the Hartmann number is zero, \( a=0 \).

References


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