Some Inequalities Related to AM-GM, Hölder and Rearrangement Inequalities

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Abstract

In this paper, we present a number of inequalities involving the AM-GM inequalities, Hölder’s inequality and rearrangement inequalities which are applicable in formation of fractional integral inequalities.

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1. Introduction

Mathematical inequalities play important role in differential and integral equations and so have applications in several fields of mathematics. While fractional
Integral inequalities are helpful to study properties, involving the existence and uniqueness of solutions, of fractional differential equations. Large dedicated literature is available to study the inequalities. We refer [2] and [3]. We also refer some recent work exploring the formation and application of inequalities; e.g. [5], [1], [6] and [4].

2. Main Results

Throughout the discussion, we denote the set of integers, positive integers, real numbers and the set of positive real numbers by the formal notations \( \mathbb{Z} \), \( \mathbb{Z}^+ \), \( \mathbb{R} \) and \( \mathbb{R}^+ \), respectively. The sequence \((a_1, a_2, a_3, \ldots, a_n)\) is denoted by \((a_j)_{j=1}^n\). We simply write \( \alpha, \beta > a \) for \( \alpha > a \), \( \beta > a \), etc. Likewise if \( x \in (a, b) \) and \( y \in (a, b) \), we write it simply by \( x, y \in (a, b) \), etc. We simple write \( a_\gamma \) for \( (a_j)_\gamma \), etc.

Let \((X, \| \|)\) be a normed linear space, \( p \) be a finite real number and for \( x_j \in X \), \( j = 1, 2, 3, \ldots \); \( x = (x_1, x_2, x_3, \ldots) \) be an infinite sequence of elements of \( X \). Then the space \( l^p(X) \), \( \|f\|_p \), \( l^\infty(X) \) and \( \|f\|_\infty \) are defined as

\[
l^p(X) = \{ x : \sum_{j=1}^{\infty} \|x_j\|^p < \infty \}, \quad \|x\|_p = \left( \sum_{j=1}^{\infty} \|x_j\|^p \right)^{1/p},
\]

\[
l^\infty(X) = \{ x : \sup_{j=1}^{\infty} \|x_j\| < \infty \}, \quad \|x\|_\infty = \sup_{j=1}^{\infty} \|x_j\|.
\]

We call the real numbers \( p \) and \( p' \) ‘Hölder’s conjugate numbers’ if the pair satisfies

\[
\frac{1}{p} + \frac{1}{p'} = 1. \tag{2.1}
\]

Theorem 2.1:

Let \( n \in \mathbb{Z}^+ \), \( p \geq \frac{1}{n} \) and \( a \geq 0 \). Then for all \( \gamma > 0 \), one has

\[
na^p \leq \frac{a^n}{p^{1-np}} + \frac{np-1}{p} \frac{1}{\gamma^p}. \tag{2.2}
\]

Proof: If \( a = 0 \), the inequality (2.2) holds as \( \frac{np-1}{p} \frac{1}{\gamma^p} \) is non-negative under the given conditions. And if \( a > 0 \), letting \( h(\gamma) = \frac{a^n}{p^{1-np}} + \frac{np-1}{p} \frac{1}{\gamma^p} \), we have
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\[ h'(\gamma) = (\gamma - a) \frac{np - 1}{p^2} \frac{1-(n+1)p}{p} \sum_{j=0}^{n-1} \gamma^{n-j} a^j, \]

which shows that \( h(\gamma) \) is minimum at \( \gamma = a \) on \((0, \infty)\). This leads to the conclusion (2.2).

**Corollary 2.1:**
Let \( n \in \mathbb{Z}^+ \), \( np \geq q \geq 0 \) and \( a \geq 0 \). For all \( \gamma > 0 \), we have

\[
na^p \leq qa^n - \frac{q-np}{p} + \frac{np-q}{p} y^p. \tag{2.3}
\]

**Proof:** If \( q = 0 \), the result (2.3) holds in the form of the identity \( n = n \). And if \( q > 0 \), replacing \( p \) by \( \frac{p}{q} \), the inequality (2.3) follows from (2.2).

**Theorem 2.2:**
If \( c_j, p_j > 0 \), \( j=1,2,3,...,n \) and

\[
\sum_{j=1}^{n} p_j = 1, \tag{2.4}
\]

then for each \( \lambda \in [-1,1] \), we have

\[
\prod_{j=1}^{n} c_j^{p_j} \leq \exp(1-\lambda^2) \sum_{j=1}^{n} c_j p_j. \tag{2.5}
\]

**Proof:** For each fixed \( \lambda \in [-1,1] \), we have

\[
\exp(\lambda x) \geq (x + \lambda) \lambda, \tag{2.6}
\]

for all \( x \in \mathbb{R} \). If \( \lambda = 0 \), (2.6) holds as \( 1 > 0 \). And if \( \lambda \neq 0 \), we take \( x = x_j \), where

\[
x_j = c_j (\lambda \sum_{j=1}^{n} c_j p_j)^{-1} - \lambda. \]

Thus, using (2.4), it follows that

\[
\prod_{j=1}^{n} ((x_j + \lambda) \lambda)^{p_j} = \frac{\prod_{j=1}^{n} c_j^{p_j}}{\sum_{j=1}^{n} c_j p_j}. \tag{2.7}
\]
\[
\prod_{j=1}^{n} \exp(\lambda p_j x_j) = \exp(1 - \lambda^2),
\]  
(2.8)

the relation (2.6) leads to the inequality

\[
\prod_{j=1}^{n} \exp(\lambda p_j x_j) \geq \prod_{j=1}^{n} ((x_j + \lambda)\lambda)^{p_j}.
\]

This leads, by use of (2.7) and (2.8), to the inequality (2.5).

Corollary 2.2:

If \( c_j, p_j > 0, \ j = 1, 2, 3, ..., n \) and \( \sum_{j=1}^{n} p_j = 1 \), then we have

\[
\prod_{j=1}^{n} c_j^{p_j} \leq \sum_{j=1}^{n} c_j p_j.
\]

(2.9)

Proof: The result follows from (2.5) taking \( \lambda \to \pm 1 \).

Example 2.1:

One may notice that for \( n = 2 \), if we take \( p_1 = p_2 = \frac{1}{2} \), we find, from (2.9), the simplest form of the AM-GM inequality

\[
\frac{c_1 + c_2}{2} \geq \sqrt{c_1 c_2},
\]

(2.10)

which is obviously valid for all \( c_1, c_2 \in R^+ \).

Moreover, it follows directly from (2.10), that for all \( c_1, c_2, c_3 \in R^+ \)

\[
c_1 c_2^2 + c_1^2 c_2 + c_1 c_3^2 + c_1^2 c_3 + c_2 c_3^2 + c_2^2 c_3 \geq 6c_1 c_2 c_3.
\]

(2.11)

Theorem 2.3:

Let \( x = (x_1, x_2, x_3, ...) \in l^p \) and \( y = (y_1, y_2, y_3, ...) \in l^{p'} \), where \( (p, p') \) is a pair of Hölder’s conjugate numbers and \( p \in [1, \infty] \), then

\[
\sum_{j=1}^{\infty} |x_j y_j| \leq \|x\|_p \|y\|_{p'}.
\]

(2.12)

Proof:

Case-1: If \( p = \infty, \ p' = 1 \) and the result (2.12) holds because
\[
\sum_{j=1}^\infty |x_j y_j| \leq \sup_{|x_j|<\infty} |x_j| \sum_{j=1}^\infty |y_j|
\]

Case 2: If \( p = 1, \ p' = \infty \) and the inequality (2.12) holds as

\[
\sum_{j=1}^\infty |x_j y_j| \leq \sup_{|x_j|<\infty} |y_j| \sum_{j=1}^\infty |x_j|
\]

(2.13)

Case 3: If \( 1 < p < \infty \), we consider the function \( f(t) = t^\theta - t\theta, \ 0 \leq t \leq 1, \ 0 < \theta < 1 \).

Then \( f'(t) = \theta(1 - t^\theta) > 0 \) on \([0,1]\), which implies that \( f \) is an increasing function on \([0,1]\). Therefore, we find that

\[
0 \leq f(t) \leq 1 - \theta.
\]

(2.14)

If we take \( \theta = \frac{1}{p}, \ t = \frac{a}{b} \), \( a, b > 0 \) then, by use of (2.1), (2.14) leads to the relation

\[
\frac{1}{a^p} \frac{1}{b^p} \leq \frac{a}{p} + \frac{b}{p'}.
\]

(2.15)

Particularly, if for each \( j \), we set \( a = (\frac{|x_j|}{\|x\|_p})^p \) and \( b = (\frac{|y_j|}{\|y\|_{p'}})^{p'} \) then (2.15) leads to the relation

\[
|x_j y_j| \leq \|x\|_p \|y\|_{p'} \left\{ \frac{1}{p} \left( \frac{|x_j|}{\|x\|_p} \right)^p + \frac{1}{p'} \left( \frac{|y_j|}{\|y\|_{p'}} \right)^{p'} \right\}.
\]

By summing over \( j \) from 1 to \( \infty \), it leads to the relation (2.12), by use of (2.1).

Corollary 2.3:
For any \( a, b > 0 \), if \((p, p')\) is a pair of Hölder’s conjugate numbers and \( p \in [1, \infty] \), then

\[
ab \leq \frac{a^p + b^{p'}}{p}. \tag{2.16}
\]

Proof: Replacing \( a \) and \( b \) by \( a^p \) and \( b^{p'} \), respectively, (2.15) leads to the desired result.

Remark 2.1:
We can also establish (2.16), using (2.9) as:

If we replace \( a, b \) and \( p \) by \( c_1^p, c_2^{p'} \) and \( \frac{m+n}{m} \), respectively, then, by use of (2.1), we find that
\[
\frac{a^n}{p} + \frac{b'^n}{p'} = c_1 \frac{m}{m+n} + c_2 \frac{n}{m+n} = c_1 p_1 + c_2 p_2, \quad p_1 = \frac{1}{p} \quad \text{and} \quad p_2 = \frac{1}{p'}.
\] (2.17)

Since (2.4) is satisfied, using \( n = 2 \) in (2.9), (2.17) leads to the relation
\[
\frac{a^n}{p} + \frac{b'^n}{p'} \geq c_1 p_{2n} = ab.
\]
Which is (2.16).

**Theorem 2.4:**

Let \((a_j^n)_{j=1}^n\) and \((b_j^n)_{j=1}^n\) be sequences of positive real numbers and \((b'_j)_{j=1}^n\) be any permutation of terms of the sequence \((b_j^n)_{j=1}^n\). If

(i) the sequences are either increasing or both decreasing, we have
\[
\sum_{j=1}^n a_j b_j \geq \sum_{j=1}^n a_j b'_j \geq \sum_{j=1}^n a_j b_{n-j+1},
\] (2.18)

(ii) one of the sequences is increasing and the other is decreasing, we have
\[
\sum_{j=1}^n a_j b_j \leq \sum_{j=1}^n a_j b'_j \leq \sum_{j=1}^n a_j b_{n-j+1}.
\] (2.19)

**Proof:** Let \( S = \sum_{j=1}^n a_j b_j \),
\[
S' = a_1b_1 + a_2b_2 + \ldots + a_{k-1}b_{k-1} + a_kb_k + a_{k+1}b_{k+1} + \ldots + a_nb_n.
\]

Then \( S - S' = (a_k - a_i)(b_k - b_i) \).

(i) If the sequences are either increasing or both decreasing, \( a_k - a_i \) and \( b_k - b_i \) have same signs. This implies that \( S \geq S' \). Repeated use of the same argument leads to the relation (2.18).

(ii) And if one of the sequences is increasing and the other is decreasing, \( a_k - a_i \) and \( b_k - b_i \) have opposite signs. So, \( S \leq S' \) and hence, repeated use of the same argument, leads to the relation (2.19).
Remark 2.2:
If \( (b_j)^n_{j=1} \) is decreasing (increasing), \( (b_n-j+1)^n_{j=1} \) is increasing (decreasing). Hence, (2.19) follows directly from (2.18). In fact, both represent the same under their conditions.

Example 2.2:
For all \( a_1, a_2, a_3 \in \mathbb{R} \), one has

\[
\sum_{j=1}^{3} a_j^2 \geq \prod_{j,k=1, j \neq k}^{3} a_j a_k .
\]  

(2.20)

Solution: If we assume that \( a_i \geq a_2 \geq a_3 \geq 0 \), the desired result (2.20) follows, for all \( a_1, a_2, a_3 \in \mathbb{R}^+ \), from (2.18) by taking \( (b_j)^3_{j=1} = (b_2, b_3, b_4) \) and \( b_j = a_j \), \( j = 1, 2, 3 \). Since the greater side of the inequality (2.20) is independent of the signs of \( a_j \)'s, the inequality (2.20) holds for all \( a_1, a_2, a_3 \in \mathbb{R} \).

Example 2.3:
For all \( a_1, a_2, a_3 \in [1, \infty) \), we have

\[
\prod_{j=1}^{3} a_j^{a_j} \geq \prod_{j=1}^{3} a_j^{a_{j+1}} .
\]  

(2.21)

Solution: Without loss of generality, we can assume that \( a_1 \geq a_2 \geq a_3 \geq 1 \), then the desired result (2.21) follows from (2.18) by taking \( (a_j)^3_{j=1} = (a_1, a_2, a_3) \), \( (b_j)^3_{j=1} = (\log a_1, \log a_2, \log a_3) \) and \( (b_j)^3_{j=1} = (\log a_2, \log a_3, \log a_1) \).

Remark 2.3:
For \( a_1 \geq a_2 \geq a_3 \geq 1 \), one may notice that \( a_1^a a_2^{-a_2} \geq a_3^a a_2^{-a_3} \) and \( a_2^a a_3^{-a_3} \geq a_3^a a_2^{-a_3} \).

Which, after multiplication, imply the result (2.21). Thus, it is an alternative way to see the same result. Now, we see a general form of the inequality (2.18) in the following lemma.

Theorem 2.5:
Let \( (a_{1,j})^n_{j=1}, (a_{2,j})^n_{j=1}, ..., (a_{m,j})^n_{j=1} \) be increasing or all decreasing sequences of positive real numbers and \( (a^*_{1,j})^n_{j=1}, (a^*_{2,j})^n_{j=1}, ..., (a^*_{m,j})^n_{j=1} \) be any permutations of terms of the sequences \( (a_{2,j})^n_{j=1}, (a_{3,j})^n_{j=1}, ..., (a_{m,j})^n_{j=1} \), respectively. Then for each \( m \in \mathbb{Z}^+ \), we have
\[
\sum_{j=1}^{n} a_j a_{2j} a_{3j} \cdots a_{mj} \geq \sum_{j=1}^{n} a_j a_{2j}^* a_{3j}^* \cdots a_{mj}^*, \quad (2.22)
\]

**Proof:** We know that if \(a \leq b\) and \(c \leq d\) for all \(a, b, c, d \in R^+\). From this fact, we induce that for each \(k \in Z^+\), \(1 < k < m\), the sequence \((a_{1,j} a_{2,j} a_{3,j} \cdots a_{k,j})_{j=1}^{n}\) is an increasing (decreasing) sequence if the individual sequences \((a_{1,j})_{j=1}^{n}\), \((a_{2,j})_{j=1}^{n}\), ..., \((a_{k,j})_{j=1}^{n}\) are all increasing (decreasing) sequences. Hence, the result \((2.22)\) follows by induction on \(m\).

**Remark 2.4:**
One may notice that the sequence \((a_{1,j} a_{2,j})_{j=1}^{n}\) may not be a monotonic sequence if one of the sequences \((a_{1,j})_{j=1}^{n}\) and \((a_{2,j})_{j=1}^{n}\) is increasing and the other one is decreasing. For example, \((a_{1,j} a_{2,j})_{j=1}^{3} = (8,14,3)\) is not a monotonic sequence while \((a_{1,j})_{j=1}^{3} = (1,2,3)\) is increasing and \((a_{2,j})_{j=1}^{3} = (8,7,1)\) is decreasing.

**Example 2.4:**
For all \(a_1, a_2, a_3 \in [0, \infty)\), one has
\[
\sum_{j=1}^{3} a_j^3 \geq 3 \prod_{j=1}^{3} a_j. \quad (2.23)
\]

**Solution:** The result \((2.23)\) follows from \((2.22)\) for \(m = 3\) by taking \((a_{1,j})_{j=1}^{3} = (a_1, a_2, a_3)\), \((a_{2,j})_{j=1}^{3} = (a_2, a_3, a_1)\), \((a_{3,j})_{j=1}^{3} = (a_3, a_1, a_2)\).

**Example 2.5:**
For all \(a_1, a_2, a_3 \in R^+\), we have
\[
\sum_{j=1}^{3} a_j^3 \geq \prod_{j=1}^{3} a_j^2 a_{j+1}. \quad (2.24)
\]

**Solution:** Using \((a_1^2, a_2^2, a_3^2)\) for \((a_j)_{j=1}^{3}\), \((b_j)_{j=1}^{3} = (a_1, a_2, a_3)\) and \((b_{j+1})_{j=1}^{3} = (a_2, a_3, a_1)\), the result \((2.24)\) follows directly from \((2.18)\).

**Theorem 2.6:**
The inequality
\[
(m+1)(a_1^{m+1} + a_2^{m+1}) \geq (a_1 + a_2) \sum_{j=0}^{m} a_j^{m-j} a_2^j
\]
holds for all...
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(i) \( a_1, a_2 \in R \), if \( m \) is an odd positive integer,
(ii) \( a_1, a_2 \in R^* \), if \( m \) is an even positive integer.

Proof: After simplification, we can rewrite (2.25) as

\[
m(a_1^{m+1} + a_2^{m+1}) \geq 2\sum_{j=1}^{m} a_1^{m-j} a_2^j. \tag{2.26}
\]

(i) If \( m \) is an odd positive integer, (2.26) can be written as

(ii) \[ m(a_1^{m+1} + a_2^{m+1}) \geq 2\sum_{j=1}^{m-1} a_1^{m+1-j} a_2^j + 2a_1^{m+1} a_2^m + 2\sum_{j=1}^{m-1} a_1^{m+1-j} a_2^j. \tag{2.27}
\]

Since the greater side of (2.27) is independent of signs of \( a_j \)'s, in order to prove the result (2.27) for all \( a_1, a_2 \in R \), it is sufficient to prove it for all \( a_1, a_2 \in R^* \) only. So, without loss of generality, we can assume that \( a_1 \geq a_2 \geq 0 \).

Taking \( (a_j)_j=1 = (a_1^{1/2}, a_2^{1/2}) = (b_j)_j=1 \) and \( (b_j^*_j=1 = (a_2^{1/2}, a_1^{1/2}) \), we find by (2.18) that

\[
a_1^{m+1} + a_2^{m+1} \geq 2a_1^{m+1} a_2^{m+1}. \tag{2.28}
\]

Then, it follows from (2.27) that we have to show only that

\[
\frac{m-1}{2} (a_1^{m+1} + a_2^{m+1}) \geq \sum_{j=1}^{m-1} a_1^{m+1-j} a_2^j + \sum_{j=1}^{m-1} a_1^{m+1-j} a_2^j. \tag{2.29}
\]

Let \( m = 2n + 1 \), \( n \in Z^+ \). Then, taking

\[
(a_j)_j=1 = (a_1^{2n+1-k}, a_2^{2n+1-k}), \quad (b_j)_j=1 = (a_1^{2n+1-k}, a_2^{2n+1-k}) \quad \text{and} \quad (b_j^*_j=1 = (a_2^{2n+1-k}, a_1^{2n+1-k}), \]

it follows, for each \( k = 0, 1, 2, ..., n-1 \), from (2.18) that

\[
a_1^{2n+2} + a_2^{2n+2} \geq a_1^{2n+1-k} a_2^{k+1} + a_2^{2n+1-k} a_1^{k+1}.
\]

That is, \( a_1^{m+1} + a_2^{m+1} \geq a_1^{m-k} a_2^{k+1} + a_2^{m-k} a_1^{k+1} \).

After adding these \( \frac{m-1}{2} \) inequalities, we find (2.29). This completes the proof of (2.25) for this case.

(iii) If \( m \) is an even positive integer, (2.26) can be expressed as

\[
m(a_1^{m+1} + a_2^{m+1})
\]
\[
\sum_{j=1}^{m-1} a_i^{m-j} a_2^j + 2 (a_1^m a_2^m + a_1^m a_2^m) + 2 \sum_{j=1}^{m-1} a_i^{m-j} a_2^{m+1+j}.
\] (2.30)

Using \( m = 2n \), \( n \in \mathbb{Z}^+ \), it becomes

\[
n(a_1^{2n+1} + a_2^{2n+1}) \geq \sum_{j=1}^{n-1} a_i^{2n+1-j} a_2^j + a_i^n a_2^n + \sum_{j=1}^{n-1} a_i^{n-j} a_2^{n+1+j}.
\] (2.31)

For any \( a_1, a_2 \in \mathbb{R}^+ \), we can assume, without loss of generality, that \( a_1 \geq a_2 \geq 0 \).

Taking \( (a_j)^2 = (a_1^{n+1}, a_2^{n+1}) \), \( (b_j)^2 = (a_1^n, a_2^n) \) and \( (b_j)^2 = (a_1^n, a_2^n) \), we find by (2.18) that

\[
a_1^{2n+1} + a_2^{2n+1} \geq a_1^{n+1} a_2^n + a_2^{n+1} a_1^n.
\] (2.32)

Then, it follows from (2.31) that we have to show only that

\[
(n-1)(a_1^{2n+1} + a_2^{2n+1}) \geq \sum_{j=1}^{n-1} a_i^{2n+1-j} a_2^j + \sum_{j=1}^{n-1} a_i^{n-j} a_2^{n+1+j}.
\] (2.33)

Taking \( (a_j)^2 = (a_1^{2n-l}, a_2^{2n-l}) \), \( (b_j)^2 = (a_1^{l+1}, a_2^{l+1}) \) and \( (b_j)^2 = (a_1^{l+1}, a_2^{l+1}) \), it follows, for each \( l = 0, 1, 2, ..., n-2 \), from (2.18) that

\[
a_1^{2n+1} + a_2^{2n+1} \geq a_1^{2n-l} a_2^{l+1} + a_2^{2n-l} a_1^{l+1}.
\]

After adding these \( n-1 \) inequalities, we find (2.33). This completes the proof of (2.25) for this case too.

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