Abstract

In 2010, Davvaz and Mirvakili [6] initiated the research on a new class of multivalued algebra which they called a Krasner \((m, n)\)-hyperring and have been able to come up with some fundamental relations on the structure. In this paper, more characterizations and some properties of the Krasner \((2, 3)\)-hyperring not explored in [6] which is called a Krasner ternary hyperring are presented. Moreover, the quotient Krasner ternary hyperring are constructed via another relation and the isomorphism theorems are proved without the normality condition.

Keywords: Krasner Ternary Hyperring, Ternary Semihyperring, Krasner \((m, n)\)-ary hyperring


1 Introduction

Hyperstructure theory was introduced in 1934 by the French Mathematician F. Marty [10] at the 8th Congress of Scandinavian Mathematicians where he defined the notion of hyperoperation on groups. In a classical algebraic structure, the binary operation of two elements of a set is again an element of the same set, while in an algebraic hyperstructure, the hyperoperation of two elements, is a subset of the same set. If this hyperoperation sends two elements to a singleton then the hyperoperation coincides with the classical binary operation.

In literature, a number of different hyperstructure theories are widely studied since these represent a suitable and natural generalization of classical algebraic structures such as groups, rings and modules and for their applications to many areas of pure and applied mathematics and computer science.

A canonical hypergroup is a nonempty set $H$ endowed with a hyperoperation “+” which is associative and commutative and having a two sided neutral element zero, a two sided inverse element and the axiom of near subtraction is satisfied, that is, $z \in x+y$ implies $x \in z-y$ and $y \in z-x$ for all $x, y, z \in H$. In this case the hyperoperation is called a canonical hyperoperation. Hyperrings are essentially rings with slightly modified operations. Rota [11] introduced a multiplicative hyperring, where the addition is a binary operation and the multiplication is a multiplicative canonical hyperoperation. If in $(R, +, \cdot)$, the “+” is the canonical hyperoperation and the “.” is a binary operation, then the hyperring is called Krasner hyperring. De Salvo [7], Assokumar and Velraj [2] studied hyperrings in which the addition and multiplication are both hyperoperations. In 2004, Davvaz [3] proved the isomorphism theorems in Krasner hyperrings provided that the hyper ideals considered in the isomorphism theorems are normal. At the same year, a book of Davvaz and Leoreanu [5] entitled “Hyperring Theory and Applications,” presented applications in applied and pure mathematics, computer science, probability, geometry, chemistry and physics.

Ternary algebra on the other hand was introduced in 1932 when D. Lehmer [8] studied certain ternary system called triplexes which turns out to be a generalization of abelian groups and of the ternary ring investigated by W.G. Lister [9] in 1971. In 2009, Davvaz [4] introduces the concept of ternary semihyperring and investigate some properties of its fuzzy hyper ideals.

In 2010, Davvaz and Mirvakili [6] defined a new class of $n$-ary multivalued algebra called an $(m,n)$-hyperring and defined a fundamental relation on the structure from which they obtained $(m,n)$-rings using the fundamental relation in [6]. Recently, Anvariyeh [1] carried out a research on $(m,n)$-ary hyper modules over $(m,n)$-hyperring and has been able to define a strongly compatible relation on the structure and determined a sufficient condition that the
said relation is transitive.

In this paper, the authors considered a particular case of the hyper structure defined in [6] by taking \( m = 2 \) and \( n = 3 \) and they called it a Krasner ternary hyperring. The structure defined in this paper can also be viewed as stronger compaired to the ternary semihyperring introduced by Davvaz [4] which also motivates the author to explore this particular case of \((m, n)\)-Krasner hyperring.

## 2 Preliminaries and Basic Definitions

**Definition 2.1.** [3] Let \( H \) be a nonempty set and let \( \wp^*(H) \) be the set of all nonempty subsets of \( H \). A map \( * : H \times H \to \wp^*(H) \) is called a hyperoperation on the set \( H \). The couple \((H, \ast)\) is called a hypergroupoid. If \( A \) and \( B \) are nonempty subsets of \( H \), we define

i. \( A \ast B = \bigcup_{a \in A, b \in B} a \ast b \);

ii. \( x \ast A = \bigcup_{a \in A} x \ast a \);

iii. \( B \ast x = \bigcup_{b \in B} b \ast x \).

**Example 2.2.** Define a hyperoperation “\( \ast \)” on \( \mathbb{N} \) by \( x \ast y = \{x, y\} \) for all \( x, y \in \mathbb{N} \). Then \((\mathbb{N}, \ast)\) is a hypergroupoid.

**Definition 2.3.** [3] A nonempty set \( H \) with a hyperoperation “\( + \)” is said to be a canonical hypergroup if the following are satisfied:

i. For every \( x, y \in H \), \( x + y = y + x \);

ii. for every \( x, y, z \in H \), \( x + (y + z) = (x + y) + z \);

iii. \( \exists 0_H \in H \) (called the neutral element of \( H \)) such that \( x + 0_H = \{x\} \) and \( 0_H + x = \{x\} \) for all \( x \in H \);

iv. for each \( x \in H \) there exists a unique element denoted by \( -x \in H \) such that \( 0_H \in [x + (-x)] \cap [(-x) + x] \);

v. for every \( x, y, z \in H \), \( z \in x + y \) implies \( y \in -x + z \) and \( x \in -y + z \).

The pair \((H, +)\) is called a canonical hypergroup.

**Example 2.4.** Let \( H = [0, 1] \). We define a hyperaddition “\( + \)” on \( H \) as follows:
\[ x + y = \begin{cases} [0, x], & \text{if } x = y, \\ \{\max\{x, y\}\}, & \text{if } x \neq y. \end{cases} \]

Then \((H, +)\) is a canonical hypergroup.

The following lemma follows directly from Definition 2.3

**Lemma 2.5.** [3] If \((H, +)\) is a canonical hypergroup, then the following hold:

i. \(- (-a) = a;\)

ii. 0 is the unique element such that for every \(a \in H\), there is an element \(-a \in H\) with the property \(0 \in a + (-a);\)

iii. \(-0 = 0;\)

iv. \(- (a + b) = -a - b;\)

for all \(a, b \in H.\)

**Lemma 2.6.** [3] Let \((H, +)\) be a canonical hypergroup and \(N \subseteq H\). Then \(N\) is a subcanonical hypergroup of \(H\) if and only if \(x - y \subseteq N\) for all \(x, y \in N.\)

**Definition 2.7.** [3] A nonempty subset \(N\) of a canonical hypergroup \(H\) is called a subcanonical hypergroup of \(H\) if \(N\) is a canonical hypergroup under the hyperoperation of \(H.\)

**Definition 2.8.** [3] A subcanonical hypergroup \(N\) of a canonical hypergroup \(H\) is said to be normal if \(x - N + x \subseteq N\) for all \(x \in H.\)

**Definition 2.9.** [9] Let \(R\) be a nonempty set. A ternary operation on \(R\) is a map \(f : R \times R \times R \rightarrow R.\)

**Example 2.10.** Consider the set \(\mathbb{Z}^-\) of negative integers and \(\cdot\) is the usual multiplication on \(\mathbb{Z}.\) Then \(\cdot\) is a ternary operation in \(\mathbb{Z}^-\) which is not binary, since \(\mathbb{Z}^-\) is not closed under the binary product.

The next definitions and results are equivalent to those found in [6] with \(m = 2\) and \(n = 3.\) These results are stated explicitly for comparison purposes and easy reference.

**Definition 2.11.** A Krasner ternary hyperring is an algebraic structure \((R, +, \cdot),\)

consisting of a nonempty set \(R,\) a hyperoperation \(\cdot\) and ternary multiplication \(\cdot\) satisfying the following:

i. \((R, +)\) is a canonical hypergroup;

ii. \((a \cdot b \cdot c) \cdot d \cdot e = a \cdot (b \cdot c \cdot d) \cdot e = a \cdot b \cdot (c \cdot d \cdot e);\)
Quotient and homomorphism in Krasner ternary hyperrings

iii. \((a + b) \cdot c \cdot d = a \cdot c \cdot d + b \cdot c \cdot d;\)

iv. \(a \cdot (b + c) \cdot d = a \cdot b \cdot d + a \cdot c \cdot d;\)

v. \(a \cdot b \cdot (c + d) = a \cdot b \cdot c + a \cdot b \cdot d;\)

vi. \(a \cdot b \cdot 0 = a \cdot 0 \cdot b = 0 \cdot a \cdot b = 0.\)

for every \(a, b, c, d, e \in R.\) If without ambiguity, \(a \cdot b \cdot c\) can be written as \(abc\) unless otherwise specified.

**Example 2.12.** Consider the canonical hypergroup \([0, 1], +\) in Example 2.4. If we define \(x \cdot y \cdot z = xyz\) for all \(x, y, z \in [0, 1],\) where juxtaposition denotes ordinary multiplication in \(\mathbb{R},\) then \(([0, 1], +, \cdot)\) is a ternary hyperring and the set \(I = \left[0, \frac{1}{2}\right]\) is a hyper ideal of \([0, 1].\)

**Definition 2.13.** Let \((R, +, \cdot)\) be a Krasner ternary hyperring. A subset \(S\) of \(R\) is called a subternary hyperring if \((S, +, \cdot)\) is itself a ternary hyperring. A subternary hyperring \(S\) of \(R\) such that \(S \neq R\) and \(S \neq \{0\}\) is called proper subternary hyperring of \(R.\)

**Definition 2.14.** A subternary hyperring \(I\) of a Krasner ternary hyperring \(R\) is called a

i. **right hyper ideal** of \(R\) if \(abi \in I\) for all \(a, b \in R\) and \(i \in I;\)

ii. **left hyper ideal** of \(R\) if \(iab \in I\) for all \(a, b \in R\) and \(i \in I;\)

iii. **lateral hyper ideal** of \(R\) if \(aib \in I\) for all \(a, b \in R\) and \(i \in I;\)

iv. **two sided hyper ideal** if \(I\) is both a right and left hyper ideal;

v. **hyper ideal** if \(I\) is left, right and lateral hyper ideal.

**Definition 2.15.** Let \((R, +, \cdot)\) be a ternary hyperring and \(A, B\) and \(C\) are nonempty subsets of \(R.\) By \(ABC,\) we mean the set

\[ABC = \bigcup \left\{ \sum_{i=1}^{n} a_ib_ic_i \mid a_i \in A, b_i \in B, c_i \in C \text{ and } n \in \mathbb{N} \right\}.\]

In particular, \(abC = \{abc \mid c \in C\},\) \(Cab = \{cab \mid c \in C\}\) and \(aCb = \{acb \mid c \in C\}.\)

**Lemma 2.16.** Let \(R\) be a Krasner ternary hyperring. Then

i. If \(I, J\) are hyper ideals of \(R,\) then \(I + J\) is a hyper ideal of \(R.\)

ii. If \(I, J,\) and \(K\) are hyper ideals of \(R,\) then \(IJK\) is a hyper ideal of \(R.\)

iii. The intersection of arbitrary hyper ideals of \(R\) is a hyper ideal of \(R.\)

iv. If \(I\) is a hyper ideal of \(R\) and \(x, y \in R,\) then \(xyI = \{xyi \mid i \in I\} = I.\)
3 Main Results

Lemma 3.1. Let \((R, +, 0)\) be a canonical hypergroup and \(x, y, z \in R\). If \(z \in x + y\), then \(-z \in -(x + y)\).

Proof: Let \(z \in x + y\). Since \((R, +)\) is a canonical hypergroup, by Definition 2.3(i) and (v), \(y \in -x + z = z + (-x)\). Applying Definition 2.3(i) and (v) again, \(-x \in -z + y = y + (-z)\) and therefore,

\[-z \in -y + (-x) = -x - y = -(x + y).\]

\[\blacksquare\]

Lemma 3.2. (Hyper ideal Criterion) Let \(R\) be a Krasner ternary hyperring. A nonempty subset \(I\) of \(R\) is a right [resp. left and lateral] hyper ideal of \(R\) if and only if for all \(i, j \in I\) and \(a, b \in R\)

i. \(i - j \subseteq I\),

ii. \(abi \in I\) [resp. \(iab \in I\) and \(aib \in I\)].

Proof: It directly follows from Lemma 2.6 that \((I, +)\) is a canonical hypergroup if and only if condition(i) is satisfied. Thus, by Definition 2.14, \(I\) is a hyper ideal of \(R\).

\[\blacksquare\]

Remark 3.3. Let \(R\) be a Krasner ternary hyperring and \(S\) a subternary hyperring of \(R\). If \(I\) is a hyper ideal of \(R\) such that \(I \subseteq S\), then \(I\) is a hyper ideal of \(S\).

To see this, since \(S\) is a subternary hyperring of \(R\), by Lemma 2.6, Theorem 3.2(i) is satisfied. On the other hand, since \(S \subseteq R\), Theorem 3.2(ii) hold. Accordingly, by Theorem 3.2, \(I\) is a hyper ideal of \(S\).

The next remark follows directly from Lemma 2.16(i) and Remark 3.3.

Remark 3.4. Let \(R\) be a Krasner ternary hyperring and \(I, J\) be right [resp. left and lateral] hyper ideal of \(R\). Then \(I \cap J\) is a hyper ideal of \(I\) and \(J\). Also, \(I\) and \(J\) are hyper ideals of \(I + J\).

Theorem 3.5. Let \(I, J\) be right [resp. left and lateral] hyper ideals of a Krasner ternary hyperring \(R\). Then \(I + J\) is the smallest hyper ideal containing \(I\) and \(J\).

Proof: By Remark 3.4, \(I \subseteq I + J\) and \(J \subseteq I + J\). Thus, \(I \cup J \subseteq I + J\). Now, let \(K\) be a hyper ideal of \(R\) such that \(I \cup J \subseteq K\) and let \(x \in I + J\). Then, there exists \(i \in I\) and \(j \in J\) such that \(x \in i + j\). Since \(K\) is a hyper ideal, by Lemma 3.2(i), \(i + j \subseteq K\). Therefore, \(x \in K\). Consequently, \(I + J \subseteq K\).
The next result is the construction of the the quotient class of a Krasner ternary hyperring where the hyper ideal considered in the construction is not necessarily normal.

**Theorem 3.6.** Let \((R,+,\cdot)\) be a Krasner ternary hyperring and \(I\) a hyper ideal of \(R\). We define the relation ‘\(\sim\)’ by \(a \sim b\) if and only if \(a \in b + I\). Then ‘\(\sim\)’ is an equivalence relation on \(R\).

**Proof:** Let \(a,b,c \in R\). Then \(0 \in a - a\). Since \(I\) is a hyper ideal of \(R\), \(0 \in I\). Therefore \(a \in \{a\} = a + 0 \subseteq a + I\). Hence \(a \sim a\). Therefore, ‘\(\sim\)’ is reflexive. Suppose \(a \sim b \in R\). Then \(a \in b + I\). Thus, \(\exists i \in I\) such that \(a \in b + i\). By Lemmas 3.1 and 2.5, \(-a \in -(b + i) = -b - i\). Thus, by Definition 2.3(iv), \(-b \in -a + i\). Consequently, by Lemmas 3.1, and 2.5(iv), \(b = -(b) \in -(a + i) = a - i\). Since \(I\) is a hyper ideal, \(-i \in I\). Hence, \(b \in a + I\). Therefore, \(b \sim a\). Accordingly, ‘\(\sim\)’ is symmetric. Finally, Suppose that \(a \sim b\) and \(b \sim c\). Then \(a \in b + I\) and \(b \in c + I\). Thus \(a \in b + i_1\) and \(b \in c + i_2\) for some \(i_1, i_2 \in I\). Hence

\[
a \in b + i_1 \subseteq \bigcup_{x \in c + i_2} x + i_1 = (c + i_2) + i_1 = c + (i_1 + i_2) \subseteq c + I.
\]

Therefore, \(a \sim c\). Accordingly, ‘\(\sim\)’ is transitive. ■

**Definition 3.7.** Let \(R\) be a ternary hyperring. The equivalence class of \(a \in R\) under ‘\(\sim\)’ is defined as \(\overline{a} = \{x \in R \mid x \sim a\}\). The set \(R/I\) is the set of all equivalence classes of \(R\).

**Remark 3.8.** Let \(R\) be a ternary hyperring, \(I\) a hyper ideal of \(R\), and \(x,y \in R\). Then \(x + I = y + I\) if and only if \(x \in y + I\).

To see this, \(\overline{x} = x + I\) and \(\overline{y} = y + I\). But \(\overline{x} = \overline{y}\) if and only if \(x \sim y\). Thus, \(x + I = y + I\) if and only if \(x \in y + I\) or \(y \in x + I\).

**Theorem 3.9.** Let \((R,+,\cdot)\) be a Krasner ternary hyperring and \(I\) a hyper ideal of \(R\). Then \(R/I\) is a Krasner ternary hyperring with the canonical hyperoperation and ternary multiplication defined as follows.

\[
(a + I) \oplus (b + I) = \{x + I \mid x \in a + b\} \text{ and } (a_1 + I) \ast (a_2 + I) \ast (a_3 + I) = a_1 a_2 a_3 + I.
\]

**Proof:** Let \(a + I, b + I, c + I, d + I \in R/I\). Then

\[
(a + I) \oplus (b + I) = \{r + I \mid r \in a + b\} = \{r + I \mid r \in b + a\} = (b + I) \oplus (a + I)
\]

Thus, ‘\(\oplus\)’ is a commutative hyperoperation on \(R/I\). Also
Thus \( \oplus \) is an associative hyperoperation on \( R/I \). Now, for every \( a + I \in R/I \),
\[
(a + I) \oplus (0 + I) = \{ x + I \mid x \in a + 0 \} = \{ x + I \mid x \in \{ a \} \} = \{ a + I \}.
\]
Hence, \( I \) is the neutral element in \( R/I \). Also, for every \( a + I \in R/I \), \( -a + I \in R/I \) and \((a + I) \oplus (-a + I) = \{ x + I \mid x \in a - a \}\). Since \( 0 \in a - a \), \( I \in (a + I) \oplus (-a + I) \). Therefore, every element of \( R/I \) has an inverse element in \( R/I \). Finally, suppose that \( c + I \in (a + I) \oplus (b + I) \). Then \( c + I \in \{ x + I \mid x \in a + b \} \). Thus \( c + I = r + I \) for some \( r \in a + b \). By Definition 2.3(v), \( a \in r - b \). So by Remark 3.8, \( c \in r + I \). It follows that there exist \( i \in I \) such that \( c \in r + i \). By Definition 2.3(v), \( r \in c - i \). Hence
\[
a \in r - b \subseteq (c - i) - b = (c - b) - i \subseteq (c - b) + I.
\]
Thus, \( a \in x + I \) for some \( x \in c - b \). By Remark 3.8, \( a + I = x + I \) for some \( x \in c - b \). So \( a + I \in \{ r + I \mid r \in c - b \} = (c + I) \oplus (-b + I) \). Similarly, \( b + I \in (c + I) \oplus (-a + I) \). Therefore, by Definition 2.3, \( (R/I, \oplus) \) is a canonical hypergroup. Thus, condition(i) of Definition 2.11 is satisfied. Finally,
\[
[(a + I) \oplus (b + I)] * (c + I) * (d + I) = \{ x + I \mid x \in a + b \} * (c + I) * (d + I) = \{ xcd + I \mid x \in a + b \} = \{ r + I \mid r \in (a + b) cd \} = \{ r + I \mid r \in acd + bcd \} = (acd + I) \oplus (bcd + I) = (a + I) * (c + I) * (d + I) \oplus (b + I) * (c + I) * (d + I).
\]
Hence, condition(ii) of Definition 2.11 is satisfied. Conditions (iii),(iv), and (v) are proved similarly. Therefore, \( (R/I, \oplus, *) \) is a Krasner ternary hyperring.
Definition 3.10. Let $I$ be a hyper ideal of a Krasner ternary hyperring $R$. The Krasner ternary hyperring $R/I$ is called the factor or quotient Krasner ternary hyperring.

Theorem 3.11. Let $I$ and $J$ be hyper ideals of a Krasner ternary hyperring $R$ such that $I \subseteq J$. Then $J/I$ is a hyper ideal of $R/I$.

Proof: Immediately follows from Remark 3.3 since $J/I \subseteq R/I$. ■

Definition 3.12. Let $(R_1,+,\cdot)$ and $(R_2,\oplus,\ast)$ be two ternary hyperrings. A mapping $\phi : R_1 \to R_2$ is called a homomorphism if the following are satisfied:

i. $\phi(a + b) = \phi(a) \oplus \phi(b)$ for all $a,b \in R_1$;

ii. $\phi(a_1 \cdot a_2 \cdot a_3) = \phi(a_1) \ast \phi(a_2) \ast \phi(a_3)$ for all $a_1,a_2,a_3 \in R_1$;

iii. $\phi(0_{R_1}) = 0_{R_2}$.

Definition 3.13. Let $\phi$ be a homomorphism from a Krasner ternary hyperring $R_1$ into a Krasner ternary hyperring $R_2$. Then

i. the set $\{x \in R \mid \phi(x) = 0\}$ is called the kernel of $\phi$ and is denoted by $\text{Ker} \phi$;

ii. the set $\{\phi(x) \mid x \in R_1\}$ is called the Image of $\phi$ and is denoted by $\text{Im} \phi$.

Lemma 3.14. If $\phi : R_1 \to R_2$ is a homomorphism, then $\phi(-x) = -\phi(x)$ for all $x \in R_1$.

Proof: Let $x \in R_1$. Since $\phi$ is a homomorphism $\phi(x - x) = \phi(x) + \phi(-x)$. Since $0 \in x - x$, it follows that $\phi(0) = 0 \in \phi(x) + \phi(-x)$. Therefore, by Definition 2.3 (iv), $\phi(-x) = -\phi(x)$. ■

Theorem 3.15. Let $\phi$ be a homomorphism from a Krasner ternary hyperring $(R_1,+,\cdot)$ into a Krasner ternary hyperring $(R_2,\oplus,\ast)$. Then

i. $\text{Ker} \phi$ is a hyper ideal of $R_1$.

ii. If $\phi$ is an epimorphism and $I$ is a hyper ideal of $R_1$, then $\phi(I)$ is a hyper ideal of $R_2$.

iii. If $B$ is a hyper ideal of $R_2$, then $f^{-1}(B) = \{a \in R_1 \mid f(a) \in B\}$ is a hyper ideal of $R_1$ containing $\text{Ker} f$.

Proof:
i. Let \( a, b \in \text{Ker} \, \phi \). Then \( \phi (a - b) = \phi (a) - \phi (b) = 0 - 0 = \{0\} \). Therefore, \( a - b \subseteq \text{Ker} \, \phi \). On the other hand, let \( a_1, a_2 \in R \) and \( i \in \text{Ker} \, \phi \). Then

\[
\phi (a_1 \cdot a_2 \cdot i) = \phi (a_1) \ast \phi (a_2) \ast \phi (i) = \phi (a_1) \ast \phi (a_2) \ast 0 = 0.
\]

Thus, \( a_1 \cdot a_2 \cdot i \in \text{Ker} \, \phi \). Therefore by Lemma 3.2, \( \text{Ker} \, \phi \) is a hyper ideal of \( R_1 \).

ii. Let \( y_1, y_2 \in \phi (I) \). Then there exist \( x_1, x_2 \in I \) such that \( \phi (x_1) = y_1 \) and \( \phi (x_2) = y_2 \). Thus, by Lemma 3.14,

\[
y_1 - y_2 = \phi (x_1) - \phi (x_2) = \phi (x_1 - x_2) \subseteq \phi (I).
\]

On the other hand, let \( y_1, y_2 \in R_2 \) and \( i \in \text{Im} \, \phi \). Since \( \phi \) is an epimorphism, there exist \( x_1, x_2, x \in R_1 \) such that \( \phi (x_1) = y_1, \phi (x_2) = y_2 \), and \( \phi (x) = i \). Thus,

\[
y_1 * y_2 * i = \phi (x_1) * \phi (x_2) * \phi (x) = \phi (x_1 \cdot x_2 \cdot x) \in \phi (I).
\]

Therefore, by Lemma 3.2 \( \phi (I) \) is a hyper ideal of \( R_2 \).

iii. Let \( a, b \in f^{-1} (B) \). Then \( f(a), f(b) \in B \). Since \( f \) is a homomorphism and \( B \) is a hyper ideal, \( f(a - b) = f(a) - f(b) \subseteq B \). Hence, \( a - b \subseteq f^{-1} (B) \). Also, for all \( i \in f^{-1} (B) \) and \( x, y \in R_1, f(xy) = f(x)f(y)f(1) \in B \). Thus, \( xyi \in f^{-1} (B) \). Similarly, \( xiy, ixy \in f^{-1} (B) \).

Therefore, by Theorem 3.2, \( f^{-1} (B) \) is a hyper ideal of \( R_1 \). Finally, if \( x \in \text{Ker} \, f \), then \( f(x) = 0 \in B \). Hence, \( x \in f^{-1} (B) \). Therefore, \( \text{Ker} \, f \subseteq f^{-1} (B) \).

**Theorem 3.16.** Let \( R \) be a Krasner ternary hyperring and \( I \) a hyper ideal of \( R \). Then the map \( \pi : R \to R/I \) defined by \( \pi (r) = r + I \) for every \( r \in R \) is an epimorphism with kernel \( I \).

**Proof:** Let \( a, b \in R_1 \) such that \( a = b \). Then \( a + I = b + I \). Thus \( \pi (a) = \pi (b) \), which means that \( \pi \) is a well defined map. Now,

\[
\pi (a + b) = \{ \pi (x) \mid x \in a + b \} \\
= \{ x + I \mid x \in a + b \} \\
= (a + I) \oplus (b + I) \\
= \pi (a) \oplus \pi (b)
\]

and

\[
\pi (abc) = abc + I \\
= (a + I)(b + I)(c + I) \\
= \pi (a) \pi (b) \pi (c).
\]
Hence, $\pi$ is a homomorphism. Also, let $x + I \in R/I$. Then $x \in R$ and $\phi(x) = x + I$. Thus $\pi$ is an epimorphism. Finally, by Remark 3.8,

\[
\text{Ker } \pi = \{ r \in R_1 \mid \pi(r) = I \} = \{ r \in R_1 \mid r + I = I \} = \{ r \in R_1 \mid r \in I \} = I
\]

The function $\pi$ in Theorem 3.16 is called the canonical ternary hyperring epimorphism.

**Theorem 3.17.** Let $f : R_1 \to R_2$ be a homomorphism of Krasner ternary hyperrings and $I$ be a hyper ideal of $R_1$ contained in the kernel of $f$. Then

i. there exists a unique homomorphism $\phi : R_1/I \to R_2$ defined by $\phi(r + I) = f(r)$;

ii. $\text{Im } f = \text{Im } \phi$;

iii. $\phi$ is an isomorphism if and only if $f$ is an epimorphism and $\text{Ker } f = I$.

**Proof:** Let $x + I, y + I, z + I \in R_1/I$ and $f : R_1 \to R_2$ be a homomorphism.

i. Suppose that $x + I = y + I$. Then, by Remark 3.8 $x \in y + I$. Thus, $x \in y + i$ for some $i \in I$. Hence,

\[
f(x) \in f(y + i) = f(y) + f(i) = f(y) + 0 = \{f(y)\}.
\]

Therefore, $f(x) = f(y)$ and so $\phi(x + I) = \phi(y + I)$. Thus, $\phi$ is a well defined map. Now,

\[
\phi((x + I) \oplus (y + I)) = \phi(\{c + I \mid c \in x + y\}) = \{f(c) \mid c \in x + y\} = f(x + y) = f(x) + f(y) = \phi(x + I) \oplus \phi(y + I)
\]

and

\[
\phi((x + I)(y + I)(z + I)) = \phi(xyz + I) = f(xyz) = f(x)f(y)f(z) = \phi(x + I)\phi(y + I)\phi(z + I)
\]

Hence, $\phi$ is a Krasner ternary hyperring homomorphism and $\phi$ is unique since it is uniquely determined by $f$. 
ii. Observe that
\[
\text{im } \phi = \{ y \in R_2 \mid \phi(x + I) = y, \exists x + I \in R/I \} \\
= \{ y \in R_2 \mid f(x) = y \} \\
= \text{im } f
\]

and
\[
\ker \phi = \{ x + I \mid \phi(x + I) = 0 \} \\
= \{ x + I \mid f(x) = 0 \} \\
= \{ x + I \mid x \in \ker f \} \\
= (\ker f)/I.
\]

iii. Suppose that \( \phi \) is an isomorphism. Then, by (ii), \( \text{im } f = \text{im } \phi = R_2 \). Thus, \( f \) is an epimorphism. On the other hand, let \( x \in \ker f \). Then \( f(x) = 0 = f(0) \). Hence, \( \phi(x + I) = \phi(0 + I) \). Since \( \phi \) is injective, \( x + I = I \). Thus, by Remark 3.8, \( x \in I \). Therefore, \( \ker f \subseteq I \). By hypothesis, \( I \subseteq \ker f \). So \( I = \ker f \). Conversely, by (i), \( \phi \) is a homomorphism. Let \( x + I, y + I \in R/I \) such that \( \phi(x + I) = \phi(y + I) \). Then \( f(x) = f(y) \). Thus, \( 0 \in f(x) - f(y) = f(x - y) \) by Definition 2.3(iv). Since \( f(0) = 0 \), there exists \( t \in x - y \) such that \( f(t) = 0 \). Hence \( t \in \ker f = I \). By Definition 2.3(v), \( x \in -(y + t) \subseteq y + I \). By Remark 3.8, \( x + I = y + I \). Therefore \( \phi \) is injective. Now, let \( y \in R_2 \). Since \( f \) is an epimorphism, \( \exists x \in R_1 \) such that \( f(x) = y \). Thus, \( \phi(x + I) = f(x) = y \). Hence \( \phi \) is surjective. Therefore, \( \phi \) is an isomorphism.

Theorem 3.18. (First Isomorphism Theorem) Let \( (R_1 +, \cdot) \) and \( (R_2, \oplus, \ast) \) be Krasner ternary hyperrings and let \( \phi : R_1 \to R_2 \) be a homomorphism. Then \( \phi \) induces an isomorphism of Krasner ternary hyperrings \( R_1/\ker \phi \cong \text{im } \phi \).

Proof: Let \( \phi : R_1 \to \text{im } \phi \). Then \( \phi \) is an epimorphism. Let \( I = \ker \phi \). By Theorem 3.17, \( R_1/\ker \phi \cong \text{im } \phi \).

Theorem 3.19. (Second Isomorphism Theorem) Let \( R \) be a Krasner ternary hyperring and \( I,J \) be hyper ideals of \( R \), then \( I/(I \cap J) \cong (I + J)/J \).

Proof: Let \( I \) and \( J \) be hyper ideals of \( R \). By Remark 3.4 and Corollary 3.5, \( I \cap J \) is a hyper ideal of \( I \) and \( J \) is a hyper ideal of \( I + J \) respectively. Thus, by Theorem 3.9, \( (I + J)/I \) and \( I/(I \cap J) \) are quotient ternary hyperrings. Define a map \( \phi : J \to (I + J)/I \) by \( \phi(j) = j + I \). Let \( a, b \in J \) such that \( a = b \).
Then $a + I = b + I$. Therefore, $\phi(a) = \phi(b)$ and $\phi$ is a well defined map. Let $a, b, c \in J$. Then

$$\phi(a + b) = \{\phi(x) \mid x \in a + b\}$$
$$= \{x + I \mid x \in a + b\}$$
$$= (a + I) \oplus (b + I)$$
$$= \phi(a) + \phi(b)$$

and

$$\phi(abc) = abc + I$$
$$= (a + I) (b + I) (c + I)$$
$$= \phi(a) \phi(b) \phi(c)$$

Therefore, $\phi$ is a homomorphism. Also

$$\ker \phi = \{x \in J \mid \phi(j) = 0\}$$
$$= \{x \in J \mid x + I = I \}$$
$$= \{x \in J \mid x \in I\}$$
$$= I \cap J$$

and

$$\text{Im} \phi = \{\phi(x) \in (J + I)/I \mid x \in J\}$$
$$= \{x + I \in (J + I)/I \mid x \in J\}$$
$$= (I + J)/I$$

Therefore, by Theorem 3.18, $I/(I \cap J) \cong (I + J)/J$. □

**Theorem 3.20. (Third Isomorphism Theorem)** Let $I$ and $J$ be hyper ideals of a Krasner ternary hyperring $R$. If $I \subseteq J$, then $R/I/J/I \cong R/J$.

**Proof:** By Theorem 3.9, $J/I$ is a hyper ideal of $R/I$. Thus by Theorem 3.11 $R/I/J/I$ is a quotient ternary hyperring. Define a map $\phi : R/I \to R/J$ by $\phi(x + I) = x + J$ and let $x + I, y + I \in R/I$ such that $x + I = y + I$. By Remark 3.8, $x \in y + I$. Thus, $x \in y + i, \exists i \in I$. Since $I \subseteq J, i \in J$. Therefore, $x \in y + i \subseteq y + J$. It follows that $x \in y + J$. Hence, by Remark 3.8, $x + J = y + J$, which means that $\phi(x) = \phi(y)$. Therefore, $\phi$ is well-defined. Let $a, b, c \in R$. Then

$$\phi[(a + I) \oplus (b + I)] = \phi\{x + I \mid x \in a + b\}$$
$$= \{\phi(x + I) \mid x \in a + b\}$$
$$= \{x + J \mid x \in a + b\}$$
$$= (a + J) \oplus (b + J)$$
$$= \phi(a + I) \oplus (b + I)$$
and

\[
\phi [(a + I)(b + I)(c + I)] = \phi (abc + I) \\
= abc + J \\
= (a + J)(b + J)(c + J) \\
= \phi (a + I) \phi (b + I) \phi (c + I)
\]

Therefore, \( \phi \) is a homomorphism. Finally,

\[
\ker \phi = \{ x + I \in R/I \mid \phi (x + I) = J \} \\
= \{ x + I \in R/I \mid x + J = J \} \\
= \{ x + I \mid x \in J \} \\
= J/I
\]

and

\[
\text{Im } \phi = \{ \phi (x + I) \mid x + I \in R/I \} \\
= \{ x + J \mid x \in R \} \\
= R/J
\]

Therefore, by Theorem 3.18, \((R/I)/J/I \cong R/J\). \(\blacksquare\)

**Theorem 3.21. (Correspondence Theorem)** If \( I \) and \( J \) are hyper ideals of a Krasner ternary hyperring \( R \), then there is a one-to-one correspondence between the set of hyper ideals of \( R \) which contains \( I \) and the set of all hyper ideals of \( R/I \), given by \( J \mapsto J/I \). Hence, every hyper ideal in \( R/I \) is of the form \( J/I \), where \( J \) is a hyper ideal of \( R \) containing \( I \).

**Proof:** Let \( I \) be a hyper ideal of \( R \). Then, by Theorem 3.16, \( f : R \to R/I \) is an epimorphism with kernel \( I \). Now, if \( J \) is a hyper ideal of \( R \), then by Theorem 3.11, \( f (J) = J/I \) is a hyper ideal of \( R/I \) and if \( K \) is a hyper ideal of \( R/I \), then by Theorem 3.15(iii), \( f^{-1} (K) \) is a hyper ideal of \( R \). Thus, the map \( \phi \) from the set of hyper ideals of \( R \) which contains \( I \) to the set of hyper ideals of \( R/I \) is a well defined map. Now, if \( K \) is a hyper ideal of \( R/I \), then \( \{ 0 + I \} = \{ I \} \subseteq K \). Thus \( I = \ker f = f^{-1} (I) \subseteq f^{-1} (K) \). Hence, \( \phi (f^{-1} (K)) = f (f^{-1} (K)) = K \) and \( \phi \) is surjective. Next, claim that if \( J \) is a hyper ideal of \( R \) containing \( I \), then \( f^{-1} (f (J)) = J \). Clearly, \( J \subseteq f^{-1} (f (J)) \). Let \( x \in f^{-1} (f (J)) \). Then, \( f (x) \in f (J) \), hence \( f (x) = f (j) \) for some \( j \in J \). Hence, \( x + I = j + I \). By Remark 3.8, \( x \in k + I \). Since \( I \subseteq J \), \( x \in k + I \subseteq J \). Thus, \( x \in J \). This proves the claim. Now, Let \( M \) and \( N \) be hyper ideals of \( R \) containing \( I \) such that \( \phi (M) = \phi (N) \). Then \( f (M) = f (N) \) and \( f^{-1} (f (M)) = f^{-1} (f (N)) \) which imply that \( M = N \) by the claim. Hence, \( \phi \) is injective. Finally, observe that \( f (J) = J/I \). \(\blacksquare\)
References


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