Invertible Extensions of Symmetric Operators and the Corresponding Generalized Resolvents

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Abstract

In this paper we study invertible extensions of a symmetric operator in a Hilbert space $H$. Generalized resolvents, which are generated by invertible self-adjoint extensions, are extracted by a boundary condition among all generalized resolvents in the Shtraus formula. It is shown that a closed symmetric invertible (not necessarily densely defined) operator in a Hilbert space has a self-adjoint invertible extension in a possibly larger Hilbert space. The corresponding extension is given by modified Shtraus’s construction, which gives additional benefits.

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1 Introduction

Let $A$ be a closed symmetric invertible operator in a Hilbert space $H$. The domain of $A$ is not supposed to be dense in $H$. Let $\tilde{A}$ be a self-adjoint extension of $A$ in a Hilbert space $\tilde{H} \supset H$. Recall that an operator-valued function $R_\lambda$, given by the following relation:

$$R_\lambda = R_\lambda(A) = R_{s,\lambda}(A) = P_H \left( \tilde{A} - \lambda E_{\tilde{H}} \right)^{-1} |_H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$
is said to be a generalized resolvent of the symmetric operator \( A \) (corresponding to the extension \( \tilde{A} \)). Starting from the fundamental papers of Neumark, Krein and Shtraus [2], [1], [3], the theory of the generalized resolvents is an object of research for many mathematicians. This theory has fruitful applications in various interpolation and moment problems and in other domains. For an exposition of Shtraus’s approach to generalized resolvents we refer to a survey paper [4]. This paper will be intensively used in the present investigation.

Fix an arbitrary point \( \lambda_0 \in \mathbb{C}\backslash\mathbb{R} \), and denote by \( \Pi_{\lambda_0} \) the upper or lower open half-plane in the complex plane, which contains \( \lambda_0 \). Recall that an arbitrary generalized resolvent \( R_{s;\lambda} \) of the operator \( A \) is given by Shtraus’s formula:

\[
R_{s;\lambda} = \begin{cases} 
(AF(\lambda) - \lambda E_H)^{-1}, & \lambda \in \Pi_{\lambda_0} \\
(AF^{*}(\overline{\lambda}) - \lambda E_H)^{-1}, & \overline{\lambda} \in \Pi_{\lambda_0}
\end{cases}
\]

where \( F(\lambda) \) is an operator-valued analytic contractive function with some special properties. Our main aim here is to characterize those generalized resolvents \( R_{s;\lambda} \), which are generated by an (at least one) invertible self-adjoint extension. Such generalized resolvents we shall call generalized I-resolvents. A generalized I-resolvent will be described by Shtraus’s formula (1), where the parameter \( F(\lambda) \) satisfies some additional boundary condition at the origin. As a main tool, we shall use a modification of Shtraus’s construction of a self-adjoint extension of a given symmetric operator (e.g., see Proof of Theorem 3.34 in [4]). As a corollary, we obtain that each closed symmetric invertible (not necessarily densely defined) operator in a Hilbert space has a self-adjoint invertible extension in a possibly larger Hilbert space.

Throughout this paper we shall follow the notations from the survey [4].

2 Invertible extensions.

Let \( A \) be a closed symmetric invertible operator in a Hilbert space \( H \). Let \( z \) from \( \mathbb{C}_- \) (\( \mathbb{C}_+ \)) be a fixed number. Recall that the following formulas (see [4, Theorem 3.13])

\[
D(B) = D(A) + (T - E_H)D(T),
\]

\[
B(f + T\psi - \psi) = Af + zT\psi - z\psi, \quad f \in D(A), \ \psi \in D(T),
\]

establish a one-to-one correspondence between all admissible with respect to \( A \) isometric operators \( T, D(T) \subseteq \mathcal{N}_z(A), R(T) \subseteq \mathcal{N}_z(A) \), and all symmetric extensions \( B \) of the operator \( A \). We have

\[
D(T) = \mathcal{N}_z(A) \cap R(B - zE_H),
\]

\[
T \subseteq (B - \overline{z}E_H)(B - zE_H)^{-1}.
\]
Formulas (2),(3) define a one-to-one correspondence between all admissible with respect to \(A\) non-expanding operators \(T\), \(D(T) \subseteq \mathcal{N}_z(A)\), \(R(T) \subseteq \mathcal{N}_z(A)\), and all dissipative (respectively accumulative) extensions \(B\) of the operator \(A\). Relations (4),(5) hold in this case, as well.

Consider the Cayley transformation of the operator \(A\):

\[
U_z = U_z(A) = (A - zE_H)(A - zE_H)^{-1} = E_H + (z - \bar{z})(A - zE_H)^{-1},
\]

The operator \(B\) may be also determined by the following relations:

\[
W_z = (B - zE_H)(B - zE_H)^{-1} = E_H + (z - \bar{z})(B - zE_H)^{-1},
\]

\[
B = (zW_z - zE_H)(W_z - E_H)^{-1} = zE_H + (z - \bar{z})(W_z - E_H)^{-1}.
\]

It is readily checked that \(\mathcal{M}_\lambda(A) = \mathcal{M}_{\frac{1}{z}}(A^{-1})\), \(\mathcal{N}_\lambda(A) = \mathcal{N}_{\frac{1}{z}}(A^{-1})\), \(\lambda \in \mathbb{R}_e\), and

\[
U_\lambda(A) = \frac{1}{\lambda} U_{\frac{1}{z}}(A^{-1}), \quad \lambda \in \mathbb{R}_e.
\]

**Theorem 2.1** Let \(A\) be a closed symmetric invertible operator in a Hilbert space \(H\), and \(z \in \mathbb{R}_e\) be a fixed point. Let \(T\) be an admissible with respect to \(A\) non-expanding operator with \(D(T) \subseteq \mathcal{N}_z(A)\), \(R(T) \subseteq \mathcal{N}_z(A)\). The following two conditions are equivalent:

(i) The operator \(B\), defined by (2) and (3), is invertible;

(ii) The operator \(\frac{1}{z} T\) is \(\frac{1}{z}\)-admissible with respect to \(A^{-1}\).

**Proof.** From relation (6) it follows that \(B\) is invertible if and only if the operator \(\frac{z}{\bar{z}} W_z\) has no non-zero fixed elements. By (7),(8) we see that

\[
\frac{z}{\bar{z}} W_z = \frac{z}{\bar{z}} U_z(A) \oplus \frac{z}{\bar{z}} T = U_{\frac{1}{z}}(A^{-1}) \oplus \frac{z}{\bar{z}} T.
\]

It remains to apply Theorem 3.6 from [4].

**Theorem 2.2** Let \(A\) be a closed symmetric invertible operator in a Hilbert space \(H\). Suppose that \(A\) has finite defect numbers. Then there exists an invertible self-adjoint operator \(\tilde{A} \supseteq A\) in a Hilbert space \(\tilde{H} \supseteq H\).
Proof. Let the operator $A$ from the formulation of the theorem have the deficiency index $(n, m)$, $n, m \in \mathbb{Z}_+$, $n + m \neq 0$. Set $\mathcal{H} = H \oplus H$ and $\mathcal{A} = A \oplus (-A)$. The closed symmetric operator $\mathcal{A}$ has equal defect numbers. Fix an arbitrary number $z \in \mathbb{R}_e$. Let $\{f_k\}_{k=1}^{n+m}$ be an orthonormal basis in $\mathcal{N}_e(A)$. Set $T(\alpha f_1) = \alpha h$, $\alpha \in \mathbb{C}$. Here $h \in \mathcal{N}_e(A)$ is an arbitrary element such that $\|h\|_{\mathcal{H}} = 1$, and $h \neq \frac{1}{\varepsilon} X_1(A^{-1})f_1$ if $f_1 \in D(X_1(A^{-1}))$; $h \neq X_1(A)f_1$ if $f_1 \in D(X_1(A))$. The operator $T$ with the domain span$\{f_1\}$ is $z$-admissible with respect to $\mathcal{A}$ and condition (iii) holds for $\mathcal{A}$. By Theorem 2.1 the symmetric extension $B$, corresponding to $T$, is invertible. The defect numbers of $B$ are equal to $n + m - 1$. If $B$ is not self-adjoint, we can take $B$ instead of $\mathcal{A}$ in the above construction to obtain a closed symmetric extension with the deficiency index $(n + m - 2, n + m - 2)$. Repeating this procedure we shall construct a self-adjoint invertible extension of $\mathcal{A}$. □

We shall strengthen the last theorem in the next section, using our result on the generalized I-resolvents.

3 Generalized I-resolvents.

Consider a closed symmetric invertible operator $A$ in a Hilbert space $H$. Choose and fix an arbitrary point $\lambda_0 \in \mathbb{R}_e$. A function $F(\lambda)$ from the set $\mathcal{S}_{\varepsilon;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A))$ (for the definition of this set see [4, p. 247]) is said to be $\lambda_0$-I-admissible with respect to the operator $A$, if the validity of

$$\lim_{\lambda \in \Pi_{\lambda_0}, \lambda \to 0} F(\lambda)\psi = \frac{\lambda_0}{\lambda_0} X_\perp(A^{-1})\psi,$$

$$\lim_{\lambda \in \Pi_{\lambda_0}, \lambda \to 0} \left[ \frac{1}{|\lambda|}(\|\psi\|_H - \|F(\lambda)\psi\|_H) \right] < +\infty,$$

for some $\varepsilon$: $0 < \varepsilon < \frac{\pi}{2}$, implies $\psi = 0$. A set of all operator-valued functions $F(\lambda) \in \mathcal{S}_{\varepsilon;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A))$ which are $\lambda_0$-I-admissible with respect to the operator $A$, we shall denote by

$$\mathcal{S}_{\varepsilon;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A)) = \mathcal{S}_{\varepsilon}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A)).$$

In the case $\mathcal{R}(A) = H$, we have $D(X_\perp(A^{-1})) = \{0\}$ and therefore the set $\mathcal{S}_{\varepsilon;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A))$ coincides with $\mathcal{S}_{\varepsilon;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A))$.

Theorem 3.1 Let $A$ be a closed symmetric invertible operator in a Hilbert space $H$, and $\lambda_0 \in \mathbb{R}_e$ be an arbitrary point. An arbitrary generalized I-resolvent $R_\lambda$ of the operator $A$ has the following form:

$$R_\lambda = \left\{ \begin{array}{ll}
(AF(\lambda) - \lambda E_H)^{-1}, & \lambda \in \Pi_{\lambda_0} \\
(AF(\lambda) - \lambda E_H)^{-1}, & \overline{\lambda} \in \Pi_{\lambda_0}
\end{array} \right.,$$

(11)
where $F(\lambda)$ is a function from $S_{I;\lambda_0}(\Pi_{\lambda_0};\mathcal{N}_{\lambda_0}(A),\mathcal{N}_{\lambda_0}(A))$. Conversely, an arbitrary function $F(\lambda) \in S_{I;\lambda_0}(\Pi_{\lambda_0};\mathcal{N}_{\lambda_0}(A),\mathcal{N}_{\lambda_0}(A))$ defines by relation (11) a generalized $I$-resolvent $R_\lambda$ of the operator $A$. Moreover, for different functions from $S_{I;\lambda_0}(\Pi_{\lambda_0};\mathcal{N}_{\lambda_0}(A),\mathcal{N}_{\lambda_0}(A))$ there correspond different generalized $I$-resolvents of the operator $A$.

**Proof.** Let $A$ be a closed symmetric invertible operator in a Hilbert space $H$, $\lambda_0 \in \mathbb{R}_e$. Let us prove the first statement of the theorem. Let $R_\lambda$ be an arbitrary generalized $I$-resolvent of the operator $A$. The generalized $I$-resolvent $R_\lambda$ is generated by a self-adjoint invertible operator $\tilde{A} \supseteq A$ in a Hilbert space $\tilde{H} \supseteq H$. Repeating the arguments from the proof of Theorem 3.34 in [4] we obtain a representation (11), where $F(\lambda)$ is a function from $S_{a;\lambda_0}(\Pi_{\lambda_0};\mathcal{N}_{\lambda_0}(A),\mathcal{N}_{\lambda_0}(A))$. It remains to check that $F(\lambda) \in S_{I;\lambda_0}(\Pi_{\lambda_0};\mathcal{N}_{\lambda_0}(A),\mathcal{N}_{\lambda_0}(A))$. The self-adjoint extension $\tilde{A}^{-1}$ of the operator $A^{-1}$ in a Hilbert space $\tilde{H} \supseteq H$ generates the following objects, see Subsection 3.6 in [4]: the operator-valued function $\mathfrak{B}_\lambda(A^{-1},\tilde{A}^{-1})$, $\lambda \in \mathbb{R}_e$; the operator $\mathfrak{B}_\infty(A^{-1},\tilde{A}^{-1})$; the operator-valued function $\mathfrak{F}(\lambda;\tilde{A}^{-1})$, $\lambda \in \Pi_{\lambda_0}$; the operator $\Phi_\infty(\tilde{A}^{-1})$. On the other hand, the operator $A$ can be identified with the operator $\mathfrak{A} = A \oplus A_e$, with $A_e = o_H$, in the Hilbert space $\tilde{H} = H \oplus H_e$, where $H_e := \tilde{H} \oplus H$, see Subsection 3.5 in [4]. Then $A^{-1} = A^{-1} \oplus A_e^{-1} = A^{-1} \oplus A_e$. By the generalized Neumann’s formulas, for $\tilde{A} \supseteq A$ there corresponds an isometric operator $T$, $D(T) = \mathcal{N}_{\lambda_0}(A)$, $R(T) = \mathcal{N}_{\lambda_0}(A)$, which is $\lambda_0$-admissible with respect to $A$. Moreover, since the operator $\tilde{A}$ is invertible, then by Theorem 2.1 we conclude that the operator $\frac{1}{\lambda_0}T$ is $\frac{1}{\lambda_0}$-admissible with respect to $A^{-1}$. Applying Theorem 3.16 in [4] for the operator $A^{-1}$, with $z = \frac{1}{\lambda_0}$, we obtain that the operator $\Phi(\frac{1}{\lambda_0};A^{-1},\frac{1}{\lambda_0}T)$ is $\frac{1}{\lambda_0}$-admissible with respect to $A^{-1}$. By the generalized Neumann’s formulas, for $\tilde{A}^{-1} \supseteq A^{-1}$ there corresponds an isometric operator $V$, $D(V) = \mathcal{N}_{\lambda_0}(A^{-1}) = \mathcal{N}_{\lambda_0}(A)$, $R(V) = \mathcal{N}_{\lambda_0}(A^{-1}) = \mathcal{N}_{\lambda_0}(A)$, which is $\frac{1}{\lambda_0}$-admissible with respect to $A^{-1}$. Consider the following operator $\mathfrak{B}(\frac{1}{\lambda_0};A^{-1},V)$, see (3.38) in [4]:

$$\mathfrak{B}(\frac{1}{\lambda_0};A^{-1},V)h = P_H^\tilde{A}^{-1}(A^{-1})Vh = P_H^\tilde{A}^{-1}h,$$

$$h \in D((A^{-1})_V) \cap H = D(\tilde{A}^{-1}) \cap H.$$

Comparing this definition with the definition of the operator $\mathfrak{B}_\infty(A^{-1},\tilde{A}^{-1})$ we conclude that $\mathfrak{B}(\frac{1}{\lambda_0};A^{-1},V) = \mathfrak{B}_\infty(A^{-1},\tilde{A}^{-1})$. Applying Theorem 3.20 in [4] for the operator $\tilde{A}^{-1}$, with $z = \frac{1}{\lambda_0}$, we get:

$$\mathfrak{B}(\frac{1}{\lambda_0};A^{-1},V) = (A^{-1})^{\Phi(\frac{1}{\lambda_0};A^{-1},V),\frac{1}{\lambda_0}}.$$
On the other hand, by the definition of $\Phi_{\infty}(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1})$ we have:

$$\mathfrak{B}_{\infty}(A^{-1}, \tilde{A}^{-1}) = (A^{-1})\Phi_{\infty}(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1}), \frac{1}{\lambda_0}.$$ 

Then

$$\Phi_{\infty}(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1}) = \Phi(\frac{1}{\lambda_0}; A^{-1}, V). \quad (12)$$

By the generalized Neumann’s formulas (4),(5) we get: $D(T) = \mathcal{N}_{\lambda_0}(A) \cap R(\tilde{A} - \lambda_0 E_{\tilde{H}}), T \subseteq (\tilde{A} - \overline{\lambda_0}E_{\tilde{H}})(\tilde{A} - \lambda_0 E_{\tilde{H}})^{-1};$

$$D(V) = \mathcal{N}_{\frac{1}{\lambda_0}}(A^{-1}) \cap R(\tilde{A}^{-1} - \frac{1}{\lambda_0}E_{\tilde{H}}) = \mathcal{N}_{\lambda_0}(A) \cap R(\tilde{A} - \lambda_0 E_{\tilde{H}}),$$

$$V \subseteq (\tilde{A}^{-1} - \frac{1}{\lambda_0}E_{\tilde{H}})(\tilde{A}^{-1} - \frac{1}{\lambda_0}E_{\tilde{H}})^{-1} = \frac{\lambda_0}{\overline{\lambda_0}}(\tilde{A} - \overline{\lambda_0}E_{\tilde{H}})(\tilde{A} - \lambda_0 E_{\tilde{H}})^{-1}.$$ 

Therefore $V = \frac{\lambda_0}{\overline{\lambda_0}}T$. By (12) we get: $\Phi_{\infty}(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1}) = \Phi(\frac{1}{\lambda_0}; A^{-1}, \frac{\lambda_0}{\overline{\lambda_0}}T)$. Therefore the operator $\Phi_{\infty}(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1})$ is $\frac{1}{\lambda_0}$-admissible with respect to $A^{-1}$. The self-adjoint operator $\tilde{A} \supseteq A$ in a Hilbert space $\tilde{H} \supseteq H$ generates the operator-valued function $\mathfrak{B}_\lambda(A, \tilde{A}, \lambda \in \mathbb{R}_e)$, and the operator-valued function $\mathfrak{B}_\lambda(\lambda; \lambda_0, A, \tilde{A}, \lambda \in \prod_{\lambda_0}$ (see [4, Subsection 3.6]). Denote (see [4, p. 222])

$$\mathfrak{B}_\lambda(A, \tilde{A}) = \left\{ h \in D(\tilde{A}) : (\tilde{A} - \lambda E_{\tilde{H}})h \in H \right\}, \lambda \in \mathbb{C}; \quad (13)$$

$$\mathfrak{B}_\lambda(A, \tilde{A}) = P_{\tilde{H}}^\perp \mathfrak{B}_\lambda, \quad \lambda \in \mathbb{C}.$$ 

We shall also need sets $\mathfrak{B}_\lambda(A^{-1}, \tilde{A}^{-1}), \mathfrak{B}_\lambda(A^{-1}, \tilde{A}^{-1}), \lambda \in \mathbb{C}$, constructed by (13) with the operators $A^{-1}, \tilde{A}^{-1}$ instead of $A, \tilde{A}$. Choose an arbitrary element $h \in \mathfrak{B}_\lambda(A, \tilde{A}), \lambda \in \mathbb{R}_e$. Then $\tilde{A}h \in R(\tilde{A})$, and

$$(\tilde{A}^{-1} - \frac{1}{\lambda}E_{\tilde{H}})\tilde{A}h = -\frac{1}{\lambda}(\tilde{A} - \lambda E_{\tilde{H}})h \in H.$$ 

Therefore $\tilde{A}h \in \mathfrak{B}_\lambda(A^{-1}, \tilde{A}^{-1})$, and $\tilde{A}\mathfrak{B}_\lambda(A, \tilde{A}) \subseteq \mathfrak{B}_\lambda(A^{-1}, \tilde{A}^{-1}), \lambda \in \mathbb{R}_e$. In order to obtain the equality: $\tilde{A}\mathfrak{B}_\lambda(A, \tilde{A}) = \mathfrak{B}_\lambda(A^{-1}, \tilde{A}^{-1}), \lambda \in \mathbb{R}_e$, it remains to apply the proved inclusion for the operators $A^{-1}, \tilde{A}^{-1}$ instead of $A, \tilde{A}$, and with $\frac{1}{\lambda}$ instead of $\lambda$.

The operator $P_{\tilde{H}}^\perp|_{\tilde{B}_\lambda(A, \tilde{A})}, \lambda \in \mathbb{R}_e$, is invertible, see considerations below (3.48) in [4]. By the definition of $\mathfrak{B}_\lambda(A, \tilde{A})$ we may write: $\mathfrak{B}_\lambda(A, \tilde{A}) = P_{\tilde{H}}^\perp\tilde{A}\left(P_{\tilde{H}}^\perp|_{\tilde{B}_\lambda(A, \tilde{A})}\right)^{-1}, \lambda \in \mathbb{R}_e$. Applying this representation for the operators $A^{-1}, \tilde{A}^{-1}$ instead of $A, \tilde{A}$, and with $\frac{1}{\lambda}$ instead of $\lambda$ we get: $\mathfrak{B}_\lambda(A^{-1}, \tilde{A}^{-1}) = \tilde{B}_\lambda(\lambda; \lambda_0, A, \tilde{A}, \lambda \in \prod_{\lambda_0}$.
By Theorem 3.32 in [4] we conclude that with respect to Invertible extensions of symmetric operators

\[ P_H^\dagger \tilde{A}^{-1} \left( P_H^\dagger |_{\mathfrak{L}_\lambda(A, \tilde{A})} \right)^{-1}, \lambda \in \mathbb{R}_e. \]

Observe that \( R(\mathfrak{B}_\lambda(A, \tilde{A})) = P_H^\dagger \tilde{A}^2_\lambda(A, \tilde{A}) = \mathfrak{L}_\lambda(A, \tilde{A}), \lambda \in \mathbb{R}_e, \) and \( \mathfrak{B}_\lambda(A, \tilde{A})g = g, g \in \mathfrak{L}_\lambda(A, \tilde{A}), \lambda \in \mathbb{R}_e. \) Therefore \( \mathfrak{B}_\lambda(A, \tilde{A})^{-1} = \mathfrak{B}_\lambda(A, \tilde{A}), \lambda \in \mathbb{R}_e. \) By the definition of the operator-valued function \( \tilde{\mathfrak{F}}(\lambda) \) and by the previous equality we may write: \( \tilde{\mathfrak{F}}(\lambda; \lambda_0, A, \tilde{A}) = (\mathfrak{B}_\lambda(A, \tilde{A}) - \lambda_0 E_H)(\mathfrak{B}_\lambda(A, \tilde{A}) - \lambda_0 E_H)^{-1}\big|_{\mathcal{N}_{\lambda_0}(A)}, \lambda \in \Pi_{\lambda_0}; \) and, also for \( \lambda \in \Pi_{\lambda_0}, \) we have:

\[
\tilde{\mathfrak{F}}\left(\frac{1}{\lambda}; \frac{1}{\lambda_0}, A^{-1}, \tilde{A}^{-1}\right) = \lambda_0 \tilde{\mathfrak{F}}(\lambda; \lambda_0, A, \tilde{A}), \quad \lambda \in \Pi_{\lambda_0}.
\] (14)

Suppose that relations (9),(10) with \( F(\lambda) = \tilde{\mathfrak{F}}(\lambda; \lambda_0, A, \tilde{A}) \) hold for some \( \varepsilon: 0 < \varepsilon < \frac{\pi}{2}. \) Then using the change of a variable \( y = \frac{1}{\lambda} \) we get:

\[
\lim_{y \in \Pi_{\lambda_0}, y \to \infty} \tilde{\mathfrak{F}}\left(\frac{1}{y}; \lambda_0, A, \tilde{A}\right) \psi = \frac{\lambda_0}{\lambda} X_{\frac{1}{\lambda_0}}(A^{-1})\psi,
\] (15)

\[
\lim_{y \in \Pi_{\lambda_0}, y \to \infty} \left| \|\psi\|_H - \|\tilde{\mathfrak{F}}\left(\frac{1}{y}; \lambda_0, A, \tilde{A}\right)\psi\|_H \right| < +\infty.
\] (16)

By (14) we obtain that \( \tilde{\mathfrak{F}}\left(\frac{1}{y}; \lambda_0, A, \tilde{A}\right) = \frac{\lambda_0}{\lambda} \tilde{\mathfrak{F}}(y; \frac{1}{\lambda_0}, A^{-1}, \tilde{A}^{-1}), \quad y \in \Pi_{\frac{1}{\lambda_0}}. \)

Substituting this expression in relations (15),(16) we get:

\[
\lim_{y \in \Pi_{\lambda_0}, y \to \infty} \tilde{\mathfrak{F}}(y; \frac{1}{\lambda_0}, A^{-1}, \tilde{A}^{-1}) \psi = X_{\frac{1}{\lambda_0}}(A^{-1})\psi,
\] (17)

\[
\lim_{y \in \Pi_{\lambda_0}, y \to \infty} \left| \|\psi\|_H - \|\tilde{\mathfrak{F}}(y; \frac{1}{\lambda_0}, A^{-1}, \tilde{A}^{-1})\psi\|_H \right| < +\infty.
\] (18)

By Theorem 3.32 in [4] we conclude that \( \psi \in D(\Phi_{\infty}(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1})), \) and \( \Phi_{\infty}(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1})\psi = X_{\frac{1}{\lambda_0}}(A^{-1})\psi. \) Since \( \Phi_{\infty}(\frac{1}{\lambda_0}; A^{-1}, \tilde{A}^{-1}) \) is \( \frac{1}{\lambda_0} \)-admissible with respect to \( A^{-1} \), we obtain that \( \psi = 0. \) Consequently, \( F(\lambda) \in \mathcal{S}_{I;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A)). \)

Let us check the second statement of the theorem. Let \( F(\lambda) \) be an arbitrary function from \( \mathcal{S}_{I;\lambda_0}(\Pi_{\lambda_0}; \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}(A)). \) We shall use the following lemma.
Lemma 3.2 Let $n, m \in \mathbb{Z}_+ \cup \{\infty\}$: $n + m \neq 0$. There exists a closed symmetric invertible operator $A$ in a Hilbert space $H$, $D(A) = H$, $R(A) = H$, which has the deficiency index $(n, m)$.

Proof. Let $H_0$ be the usual Hilbert space $l^2$ of all complex sequences $\xi = (\xi_k)_{k=0}^\infty$, such that $\|\xi\|_2^2 = \sum_{k=0}^\infty |\xi_k|^2 < \infty$. Let $\mathfrak{A} = \{f_k\}_{k=0}^\infty$ be an orthonormal basis in $H_0$, with $f_k = (\delta_{j,k})_{j=0}^\infty$. Consider the following operator (unilateral shift): $V_0 h = \sum_{k=0}^\infty \alpha_k f_{k+1}, \quad h = \sum_{k=0}^\infty \alpha_k f_k \in H_0, \alpha_k \in \mathbb{C}$, with $D(V_0) = H_0$. The operator $V_0$ is closed, isometric, and its deficiency index is $(0, 1)$. The condition $V_0 g = \pm g$, for an element $g \in H_0$, implies $g = 0$. Consequently, the inverse Cayley transformation: $A_0 = i(V_0 + E_{H_0})(V_0 - E_{H_0})^{-1},$ is a closed symmetric invertible operator in $H_0$, with the deficiency index $(0, 1)$. If $h \in H_0$ and $h \perp (V_0 \pm E_{H_0})H_0$, then $V_0^* h = \mp h$. The condition $V_0^* h = \mp h$ implies $h = 0$. Therefore, $\overline{D(A_0)} = H_0, \overline{R(A_0)} = H_0$. \(\overline{H_0} = \bigoplus_{j=0}^\infty H_0,\)

\(W_l = \bigoplus_{j=0}^l V_0, \quad l \in \mathbb{Z}_+ \cup \{\infty\}.\) $W_l$ is a closed isometric operator in $H_l$. The deficiency index of $W_l$ is equal to $(0, l + 1)$. If $W_l h \pm \mp h$, or $W_l^* h = \mp h$, then $h = 0$. Then $A_l = i(W_l + E_{H_l})(W_l - E_{H_l})^{-1},$ is a closed symmetric invertible operator in $H^l$, with the deficiency index $(0, l + 1), l \in \mathbb{Z}_+ \cup \{\infty\}$. Moreover, we have $\overline{D(A_l)} = H^l, \overline{R(A_l)} = H^l$. Observe that the operator $-A_l$ has the deficiency index $(l + 1, 0), l \in \mathbb{Z}_+$. If $m > 0, n > 0$, we set $H = H^{m-1} \oplus H^{n-1}, \quad A = (-A_{m-1}) \oplus A_{n-1}$. \(\square\)

Let us return to the proof of the theorem. If the operator $A$ is self-adjoint, then the set $\mathcal{S}_{l, \lambda_0}(\Pi_{\lambda_0}, \mathcal{N}_{\lambda_0}(A), \mathcal{N}_{\lambda_0}^\perp(A))$ consists of a unique function $F(\lambda) = \sigma_{H^l}$. Of course, this function generates the resolvent of $A$ by (11). Thus, we can assume that $A$ is not self-adjoint.

We shall use the scheme of the proof of the corresponding statement in Theorem 3.34 in [4]. By Lemma 3.2 there exists a closed symmetric invertible operator $A_1$ in a Hilbert space $H_1$, $\overline{D(A_1)} = H_1, \overline{R(A_1)} = H_1$, which has the same defect numbers as $A$. Let $U$ and $W$ be arbitrary isometric operators, which map respectively $\mathcal{N}_{\lambda_0}(A_1)$ on $\mathcal{N}_{\lambda_0}(A)$, and $\mathcal{N}_{\lambda_0}^\perp(A_1)$ on $\mathcal{N}_{\lambda_0}^\perp(A)$. Set $F_1(\lambda) = W^{-1}F(\lambda)U, \lambda \in \Pi_{\lambda_0}$. Since $F_1(\lambda)$ belongs to $\mathcal{S}(\Pi_{\lambda_0}, \mathcal{N}_{\lambda_0}(A_1), \mathcal{N}_{\lambda_0}^\perp(A_1))$, by Shtraus's formula it generates a generalized resolvent $\mathfrak{R}_\lambda(A_1)$ of $A_1$. Let us check that $\mathfrak{R}_\lambda(A_1)$ is generated by a self-adjoint invertible operator $\widehat{A}_1 \supseteq A_1$ in a Hilbert space $\widehat{H}_1 \supseteq H_1$. Suppose that $\mathfrak{R}_\lambda(A_1)$ is generated by a self-adjoint operator $\widehat{A}_1 \supseteq A_1$ in a Hilbert space $\widehat{H}_1 \supseteq H_1$. Suppose that $U_i(\widehat{A}_1) h = -h$, for an element $h \in \widehat{H}_1$. Then \((h, U_i(\widehat{A}_1) g)_{\widehat{H}_1} = -(U_i(\widehat{A}_1) h, U_i(\widehat{A}_1) g)_{\widehat{H}_1} = -(h, g)_{\widehat{H}_1}, \quad g \in \widehat{H}_1, \quad 0 = (h, (U_i(\widehat{A}_1) + E_{\widehat{H}_1}) g)_{\widehat{H}_1}, \quad g \in \widehat{H}_1.\) In particular, $h$ is orthogonal to $(U_i(A_1) + E_{H_1})D(U_i(A_1)) = R(A_1)$. Then $h \in \widehat{H}_1 \ominus H_1$. Set $\widehat{H}_0 = \{h \in \widehat{H}_1: U_i(\widehat{A}_1) h = -h\}$. Observe that $\widehat{H}_0$ is a subspace of
\( \tilde{H}_1 \oplus H_1 \). Then \( \tilde{H}_1 = H_1 \oplus (\tilde{H}_1 \oplus H_1) = H_1 \oplus ( (\tilde{H}_1 \oplus H_1) \oplus \tilde{H}_0 ) \oplus \tilde{H}_0 = \hat{H}_1 \oplus \tilde{H}_0 \), where \( \hat{H}_1 := H_1 \oplus \left( (\tilde{H}_1 \oplus H_1) \oplus \tilde{H}_0 \right) \). Notice that \( U_s(\tilde{A}_1)\tilde{H}_0 = \hat{H}_0 \) and \( U_s(\tilde{A}_1)\tilde{H}_1 = \tilde{H}_1 \). Set \( W_1 = U_s(\tilde{A}_1)|_{\tilde{H}_1} \). There are no non-zero elements \( g \in \tilde{H}_1 \) such that \( W_1 g = -g \). Then the inverse Cayley transformation \( \tilde{A}_1 := i(W_1 + E_{\tilde{H}_1})(W_1 - E_{\tilde{H}_1})^{-1} \), is invertible. Since \( \tilde{A}_1 \subseteq \tilde{A}_1 \), then \( (\tilde{A}_1 - \lambda E_{\tilde{H}_1})^{-1} \subseteq (\tilde{A}_1 - \lambda E_{\tilde{H}_1})^{-1} \). Therefore \( \tilde{A}_1 \) generates \( R_\lambda(A_1) \). Set \( H_e := \tilde{H}_1 \oplus H_1 \). Starting from the same formula, we repeat the rest of the arguments in the proof of Theorem 3.34 in [4]. In what follows, we shall use notations and constructions from this proof without additional references. We shall obtain a self-adjoint operator \( \tilde{A} \supseteq A \) in a Hilbert space \( \hat{H} \supseteq H \), which generates a generalized resolvent \( R_\lambda \) of \( A \). This generalized resolvent is related to \( F(\lambda) \) by (11). It remains to check that the operator \( \tilde{A} \) is invertible. Since the operator \( \tilde{A}_1 \) is invertible, then by Theorem 2.1 we obtain that the operator \( \lambda T \) is \( \frac{1}{\lambda_0} \)-admissible with respect to \( A_1^{-1} \). By Theorem 3.16 in [4] we conclude that the operators \( \Phi(\frac{1}{\lambda_0}; A_1^{-1}, \lambda_0 T) \) and \( \lambda_0 T_{22} \) are \( \frac{1}{\lambda_0} \)-admissible with respect to \( A_1^{-1} \) and \( A_e^{-1} = A_0 \), respectively. Comparing the domains of \( \Phi(\frac{1}{\lambda_0}; A_1^{-1}, \lambda_0 V) \) and \( \Phi(\frac{1}{\lambda_0}; A_e^{-1}, \lambda_0 V) \) we conclude that

\[
D(\Phi(\frac{1}{\lambda_0}; A^{-1}, \lambda_0 V)) = U D(\Phi(\frac{1}{\lambda_0}; A_1^{-1}, \lambda_0 T)).
\]

Using Remark 3.15 and formula (3.28) in [4, p. 218] for \( \Phi(\frac{1}{\lambda_0}; A^{-1}, \lambda_0 V) \) and \( \Phi(\frac{1}{\lambda_0}; A_1^{-1}, \lambda_0 V) \), we get: \( \Phi(\frac{1}{\lambda_0}; A^{-1}, \lambda_0 V) = W \Phi(\frac{1}{\lambda_0}; A_1^{-1}, \lambda_0 T) U^{-1} \). We can apply the arguments in the proof of the already proved first statement of the theorem for the operator \( A := A_1 \); the point \( \lambda_0 \); the generalized \( I \)-resolvent \( R_\lambda := R_\lambda(A_1) \) of \( A_1 \), which is generated by the self-adjoint invertible operator \( \tilde{A} := \tilde{A}_1 \) in \( \tilde{H}_1 \supseteq H_1 \).

\[
\Phi_\infty(\frac{1}{\lambda_0}; A_1^{-1}, \tilde{A}_1^{-1}) = \Phi(\frac{1}{\lambda_0}; A_1^{-1}, \lambda_0 T).
\]

\[
\mathcal{F}(\frac{1}{\lambda}; \lambda_0, A_1^{-1}, \tilde{A}_1^{-1}) = \frac{\lambda_0}{\lambda_0} \mathcal{F}(\lambda; \lambda_0, A_1, \tilde{A}_1), \quad \lambda \in \Pi_{\lambda_0}.
\]

Then

\[
\mathcal{F}(y; \lambda_0, A_1^{-1}, \tilde{A}_1^{-1}) = \frac{\lambda_0}{\lambda_0} \mathcal{F}(\frac{1}{y}; \lambda_0, A_1, \tilde{A}_1), \quad y \in \Pi_{\lambda_0}.
\]

By Theorem 3.32 in [4] we have:

\[
D(\Phi_\infty(\frac{1}{\lambda_0}; A_1^{-1}, \tilde{A}_1^{-1})) = \{ \psi \in \mathcal{N}_{\lambda_0}(A_1^{-1}) : \}.
\]
\[
\lim_{\lambda \in \mathcal{P}_{{\lambda}_0}, \lambda \to \infty} \left[ |\lambda| (\|\psi\|_{H_1} - \|\mathcal{F}(\lambda; {\frac{1}{{\lambda}_0}}, A_1^{-1}, \tilde{A}_1^{-1})\psi\|_{H_1}) \right] < +\infty \right),
\]

\[
\Phi_{\infty}(\frac{1}{{\lambda}_0}; A_1^{-1}, \tilde{A}_1^{-1})\psi = \lim_{\lambda \in \mathcal{P}_{{\lambda}_0}, \lambda \to \infty} \mathcal{F}(\lambda; {\frac{1}{{\lambda}_0}}, A_1^{-1}, \tilde{A}_1^{-1})\psi,
\]

\[
\psi \in D(\Phi_{\infty}(\frac{1}{{\lambda}_0}; A_1^{-1}, \tilde{A}_1^{-1})), \text{ where } 0 < \varepsilon < \frac{\pi}{2}.
\]

Using (19),(20),(23),(24),(22) and the change of a variable: \( y = \frac{1}{\lambda} \), we obtain that \( D(\Phi(\frac{1}{{\lambda}_0}; A^{-1}, \frac{{\lambda}_0}{\lambda} V)) = \{ \psi \in \mathcal{N}_{\lambda_0}(A) : \)

\[
\lim_{y \in \mathcal{P}_{{\lambda}_0}, y \to 0} \left[ \frac{1}{|y|} (\|\psi\|_H - \|F(y)\psi\|_H) \right] < +\infty \right);
\]

\[
\Phi(\frac{1}{{\lambda}_0}; A^{-1}, \frac{{\lambda}_0}{\lambda} V)\psi = \frac{{\lambda}_0}{\lambda} \lim_{y \in \mathcal{P}_{{\lambda}_0}, y \to 0} F(y)\psi, \quad \psi \in D(\Phi(\frac{1}{{\lambda}_0}; A^{-1}, \frac{{\lambda}_0}{\lambda} V)).
\]

Suppose that there exists an element \( \psi \in D(\Phi(\frac{1}{{\lambda}_0}; A^{-1}, \frac{{\lambda}_0}{\lambda} V)) \cap X_{\frac{1}{{\lambda}_0}}(A^{-1}) \) such that the following equality holds: \( \Phi(\frac{1}{{\lambda}_0}; A^{-1}, \frac{{\lambda}_0}{\lambda} V)\psi = X_{\frac{1}{{\lambda}_0}}(A^{-1})\psi \). By (25),(26) this means that \( \psi \in \mathcal{N}_{\lambda_0}(A) \) and \( \lim_{y \in \mathcal{P}_{{\lambda}_0}, y \to 0} [\frac{1}{|y|} (\|\psi\|_H - \|F(y)\psi\|_H)] < +\infty \), \( \frac{{\lambda}_0}{\lambda} \lim_{y \in \mathcal{P}_{{\lambda}_0}, y \to 0} F(y)\psi = X_{\frac{1}{{\lambda}_0}}(A^{-1})\psi \). Since \( F(\lambda) \) is \( \lambda_0 \)-admissible with respect to the operator \( A \), we get \( \psi = 0 \). This means that \( \Phi(\frac{1}{{\lambda}_0}; A^{-1}, \frac{{\lambda}_0}{\lambda} V) \) is \( \frac{1}{{\lambda}_0} \)-admissible with respect to \( A^{-1} \). Since \( \frac{{\lambda}_0}{\lambda} T_{22} \) is \( \frac{1}{{\lambda}_0} \)-admissible with respect to \( A_e \), then by Theorem 3.16 in [4] we obtain that the operator \( \frac{{\lambda}_0}{\lambda} V \) is \( \frac{1}{{\lambda}_0} \)-admissible with respect to \( A^{-1} \). By Theorem 2.1 we conclude that the operator \( A_V = \tilde{A} \) is invertible. The last statement of the theorem follows directly from Shtraus’s formula. \( \square \)

**Corollary 3.3** Let \( A \) be a closed symmetric invertible operator in a Hilbert space \( H \). Then there exists an invertible self-adjoint operator \( \tilde{A} \supseteq A \) in a Hilbert space \( \tilde{H} \supseteq H \).

**Proof.** Choose an arbitrary point \( \lambda_0 \in \mathbb{R}_e \). Set \( F(\lambda) = 0_{\mathcal{N}_{\lambda_0}(A)} \). If relation (9) holds, then \( 0 = \frac{{\lambda}_0}{\lambda} X_{\frac{1}{{\lambda}_0}}(A^{-1})\psi \). Since the forbidden operator is isometric, we get \( \psi = 0 \). In a similar manner, if the following relation holds: \( \lim_{\lambda \in \mathcal{P}_{{\lambda}_0}, \lambda \to \infty} F(\lambda)\psi = X_{\lambda_0}(A)\psi \), then \( 0 = X_{\lambda_0}(A)\psi \), and \( \psi = 0 \). Consequently, the operator-valued function \( F(\lambda) \) is \( \lambda_0 \)-admissible with respect to \( A \). By Theorem 3.1 we obtain that the set of generalized \( I \)-resolvents is not empty, see also the construction of \( \tilde{A} \) in the proof of that theorem. \( \square \)
References


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