Nonlinear Control Method of Chaos Synchronization
for Arbitrary 2D Quadratic Dynamical Systems
in Discrete-Time

Adel Ouannas

Department of Mathematics and Computer Science
University of Tebessa, Algeria

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Abstract
In this paper, an arbitrary full state hybrid projective synchronization (AFSHPS) for general discrete quadratic chaotic systems is investigated. Based on Lyapunov stability theory and nonlinear control method, a new controller is designed for achieving the AFSHPS for 2D quadratic chaotic systems in discrete-time. Numerical simulation validate the main result of this paper.

Keywords: Arbitrary full state hybrid projective synchronization, discrete-time, quadratic systems, chaos, nonlinear control

1 Introduction
Chaos synchronization in discrete-time has been extensively studied, due to its potential applications for secure communication [1, 2, 3]. Different types and various powerful methods and techniques of chaos synchronization have been reported to investigate chaos synchronization in discrete dynamical systems [4, 5, 6, 7, 8, 9].

Recently, a novel type of discrete chaos synchronization, known as arbitrary full state hybrid projective synchronization, has been introduced and applied to chaotic systems [10], which includes projective synchronization, hybrid projective synchronization. In AFSHPS, each master system state synchronizes
with a linear combination of slave system states and the scaling constants are chosen arbitrary. In this paper, an AFSHPS scheme is proposed, for 2D quadratic chaotic systems in discrete-time, using new synchronization method. To verify the effectiveness of the proposed approach, the new scheme is applied between the drive Lorenz discrete-time system [9] and the controlled response system Hénon map,[11].

This paper is organized as follows. In section 2, definition of AFSHPS is introduced. In section 3, description of drive-response chaotic systems addressed in this paper are provided. In section 4, a new synchronization method is proposed. In section 5, numerical example is used to verify the effectiveness of the proposed scheme. Finally, conclusion is given in section 6.

2 Definition of AFSHPS

We consider the chaotic systems in the following forms

\[ X(k + 1) = F(X(k)), \]
\[ Y(k + 1) = G(Y(k)) + U, \]

where \( X(k) = (x_1(k), x_2(k), ..., x_n(k))^T \), \( Y(k) = (y_1(k), y_2(k), ..., y_n(k))^T \) are the state vectors, \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( G : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( U \in \mathbb{R}^n \) is the controller to be determined. Those two systems are in arbitrary full-state hybrid projective synchronization (AFSHPS) when, for an initial condition, there is a matrix \( \Lambda = (\Lambda_{ij}) \in \mathbb{R}^{n \times n} \) such that

\[ \lim_{k \to \infty} \| e(k) \| = \lim_{k \to \infty} \| Y(k) - \Lambda X(k) \| = 0, \]

where \( e(k) \) is the errors dynamical systems and \( \| . \| \) is the euclidean norm.

3 Drive-response systems description

We consider the drive system described by the following quadratic chaotic system

\[
\begin{align*}
    x_1(k + 1) &= \sum_{j=1}^{2} a_{1j} x_1(k) + X^T(k) C^1 X(k) + \alpha_1, \\
    x_2(k + 1) &= \sum_{j=1}^{2} a_{2j} x_1(k) + X^T(k) C^2 X(k) + \alpha_2,
\end{align*}
\]

where \( X(k) = (x_1(k), x_2(k))^T \in \mathbb{R}^2 \), is the state vector of drive system, \( (a_{ij}) \in \mathbb{R}^{2 \times 2} \), \( C^i = (c_{ij}) \in \mathbb{R}^{2 \times 2} \) \( (i = 1, 2) \) and \( (\alpha_1, \alpha_2) \in \mathbb{R}^2 \) are real numbers. As the response system, we consider the following quadratic chaotic system described by
Nonlinear control method of chaos synchronization

\[ \begin{align*}
  y_1(k+1) &= \sum_{j=1}^{2} b_{1j} y_j(k) + Y^T(k) D^1 Y(k) + \beta_1 + u_1, \\
  y_2(k+1) &= \sum_{j=1}^{2} b_{2j} y_j(k) + Y^T(k) D^2 Y(k) + \beta_2 + u_2,
\end{align*} \tag{5} \]

where \( Y(k) = (y_1(k), y_2(k))^T \in \mathbb{R}^2 \) is the state vector of response system, \( (b_{ij}) \in \mathbb{R}^{2 \times 2} \), \( D^i = (d^i_{1j}) \in \mathbb{R}^{2 \times 2} \quad (i = 1, 2) \), \( (\beta_1, \beta_2) \in \mathbb{R}^2 \) are real numbers and \( (u_1, u_2)^T \in \mathbb{R}^2 \) is the vector controller to be determined.

4 Synchronization method

The AFSHPS errors between the drive system (4) and the response system (5) are defined by

\[ \begin{align*}
  e_1(k+1) &= y_1(k+1) - \Lambda_{11} x_1(k+1) - \Lambda_{12} x_2(k+1), \\
  e_2(k+1) &= y_2(k+1) - \Lambda_{21} x_1(k+1) - \Lambda_{22} x_2(k+1),
\end{align*} \tag{6} \]

where \( (\Lambda_{ij}) \in \mathbb{R}^{2 \times 2} \) are arbitrary scaling constants. The errors dynamics between (4) and (5), can be derived as

\[ \begin{align*}
  e_1(k+1) &= \sum_{j=1}^{2} b_{1j} e_j(k) + R_1 + f_1 + g_1 + u_1, \\
  e_2(k+1) &= \sum_{j=1}^{2} b_{2j} e_j(k) + R_2 + f_2 + g_1 + u_1,
\end{align*} \tag{7} \]

where

\[ \begin{align*}
  R_1 &= \sum_{j=1}^{2} (a_{1j} - \Lambda_{11} b_{1j}) x_j(k) + \sum_{j=1}^{2} (a_{1j} - \Lambda_{11} b_{1j}) x_j(k), \\
  R_2 &= \sum_{j=1}^{2} (a_{2j} - \Lambda_{21} b_{2j}) x_j(k) + \sum_{j=1}^{2} (a_{2j} - \Lambda_{22} b_{2j}) x_j(k),
\end{align*} \tag{8} \]

\[ \begin{align*}
  f_1 &= Y^T(k) D^1 Y(k) - \Lambda_{11} X^T(k) C^1 X(k) - \Lambda_{12} X^T(k) C^2 X(k), \\
  f_2 &= Y^T(k) D^2 Y(k) - \Lambda_{21} X^T(k) C^1 X(k) - \Lambda_{22} X^T(k) C^2 X(k),
\end{align*} \tag{9} \]

and

\[ \begin{align*}
  g_1 &= -\Lambda_{11} \alpha_1 - \Lambda_{12} \alpha_2 + \beta_1, \\
  g_2 &= -\Lambda_{21} \alpha_1 - \Lambda_{22} \alpha_2 + \beta_2.
\end{align*} \tag{10} \]

To achieve synchronization between systems (4) and (5), we choose the vector controller as:

\[ \begin{align*}
  u_1 &= (b_{21} - l_1) e_1(k) - (b_{22} + 2b_{12} - l_2) e_2(k) - R_1 - f_1 - g_1, \\
  u_2 &= (b_{11} - l_1) e_1(k) + (b_{12} - l_2) e_2(k) - R_2 - f_2 - g_1,
\end{align*} \tag{11} \]

where \( (l_1, l_2) \in \mathbb{R}^2 \) are constants to be determined. Substituting Eq. (11) into Eq. (7), one can simply the synchronization errors into:
\[
\begin{align*}
  e_1 (k + 1) &= (b_{11} + b_{21} - l_1) e_1 (k) - (b_{22} + b_{12} - l_2) e_2 (k), \\
  e_2 (k + 1) &= (b_{11} + b_{21} - l_1) e_1 (k) + (b_{22} + b_{12} - l_2) e_2 (k).
\end{align*}
\] (12)

**Theorem 1** If \(l_1\) and \(l_2\) are chosen such that the following inequalities:

\[
|b_{2j} + b_{1j} - l_j| < \frac{1}{\sqrt{2}}, \quad (j = 1, 2),
\] (13)

holds. Then, the two systems (4) and (5), are globally synchronized.

**Proof.** We take as a candidate Lyapunov function:

\[
V (e (k)) = e_1^2 (k) + e_2^2 (k),
\] (14)

we get:

\[
\Delta V (e (k)) = V (e (k + 1)) - V (e (k))
= 2 \sum_{i=1}^{2} e_i^2 (k + 1) - 2 \sum_{i=1}^{2} e_i^2 (k)
= \sum_{j=1}^{2} \left( 2 (b_{2j} + b_{1j} - l_j)^2 - 1 \right) e_j^2 (k)
  + 2 [(b_{11} + b_{21} - l_1) (b_{22} + b_{12} - l_2) - (b_{11} + b_{21} - l_1) (b_{22} + b_{12} - l_2)] e_1 (k) e_2 (k).
\]

By using (13), we obtain: \(\Delta V (e (k)) < 0\). Thus, by Lyapunov stability it is immediate that \(\lim_{k \to \infty} e_i (k) = 0\), \((i = 1, 2)\), and from the fact \(\lim_{k \to \infty} \|e (k)\| = 0\). We conclude that the systems (4) and (5) are globally synchronized. \(\blacksquare\)

5 Numerical example

To demonstrate the use of chaos synchronization criterion proposed herein, an example of chaotic systems is considered in this section.

The drive system is described by the 2D Lorenz discrete-time system [9].

\[
\begin{align*}
  x_1 (k + 1) &= (1 + \alpha \beta) x_1 (k) - \beta x_2 (k) x_1 (k), \\
  x_2 (k + 1) &= (1 - \beta) x_2 (k) + \beta x_1^2 (k),
\end{align*}
\] (15)

where \(x_1 (k)\), \(x_2 (k)\) are the states and \(a\), \(b\) are bifurcation parameters of the system. The 2D Lorenz discrete-time system is chaotic when \(\alpha = 1.25\) and \(\beta = 0.75\). Figure 1 illustrate the chaotic attractor of system

The response system is described by the controlled Hénon map [11].

\[
\begin{align*}
  y_1 (k + 1) &= y_2 (k) - a y_1^2 (k) + 1 + u_1, \\
  y_2 (k + 1) &= b y_1 (k) + u_2,
\end{align*}
\] (16)
Figure 1: Chaotic attractor of Lorenz discrete-time system

where $y_1(k)$, $y_2(k)$ are the states, $a$, $b$ are bifurcation parameters of the controlled system and $u_1, u_2$ are synchronization controllers. The Hénon map is chaotic when $a = 1.4$ and $b = 0.3$.

Figure 2 illustrate the chaotic attractor of system (16).

Corollary 2 For the two coupled Lorenz discrete-time system and Hénon map, if $(l_1, l_2)$ are chosen such that the following inequalities:

$$
\begin{align*}
-0.4 < l_1 < 1, \\
0.3 < l_2 < 1.7,
\end{align*}
$$

holds. Then, they are globally synchronized.
Finally, if we choose \((l_1, l_2) = (-0.06, 0.75)\), the conditions of Theorem 1 are satisfied and by using Matlab we get the numeric result that is showed in Fig. 3.

![Figure 3: Synchronization errors between Lorenz discrete-time system and Henon map](image)

### 6 Conclusion

In this paper, to achieve arbitrary full state hybrid synchronization (AFSHPS), a new nonlinear control method was presented for two coupled of arbitrary 2D quadratic dynamical systems in discrete-time. It was shown that the proposed synchronization criterion was based on a simple and effective result. The numerical example was utilized to illustrate the effectiveness of the proposed method.

### References


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