Fair Total Domination in the Join, Corona, and Composition of Graphs

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Abstract

In this paper, we characterize the fair total dominating sets in the join and corona of graphs and determine the corresponding fair total domination numbers. We also characterize some fair total dominating sets in the composition of graphs and give sharp upper bounds for the corresponding fair total domination numbers.

Keywords: Fair total domination; Composition of graphs

1 Introduction

Let \( G = (V(G), E(G)) \) be a graph and \( v \in V(G) \). The neighborhood of \( v \) is the set \( N_G(v) = N(v) = \{ u \in V(G) : uv \in E(G) \} \). If \( X \subseteq V(G) \), the
open neighborhood of $X$ is the set $N_G(X) = N(X) = \cup_{v \in X} N_G(v)$. The closed neighborhood of $X$ is $N_G[X] = N[X] = X \cup N(X)$. The degree of a vertex $v \in V(G)$, denoted by $\deg_G(v)$, is equal to the cardinality of $N_G(v)$.

A set $C \subseteq V(G)$ is a dominating set of $G$ if for every $v \in V(G) \setminus C$, there exists $u \in C$ such that $uv \in E(G)$, that is, $N_G[C] = V(G)$. The domination number of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of $G$. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. This concept was introduced by Claude Berge [7]. Moreover, Oystein Ore [6] introduced the terms dominating set and domination number in his book on graph theory which was published in 1962. A set $S \subseteq V(G)$ is a total dominating set of $G$ if for every $v \in V(G)$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G(S) = V(G)$. This concept was introduced by Cockayne, Dawes, and Hedetniemi [5].

Let $G$ be a graph that is not the empty graph. For an integer $k \geq 1$, a $k$-fair dominating set, abbreviated $k$FD-set, is a dominating set $S \subseteq V(G)$ such that $|N(u) \cap S| = k$ for every $u \in V(G) \setminus S$. The $k$-fair domination number of $G$, denoted $\gamma_{kfd}(G)$, is the minimum cardinality of a $k$FD-set. A $k$-fair dominating set of cardinality $\gamma_{kfd}(G)$ is called a $\gamma_{kfd}(G)$-set.

A fair dominating set (FD-set) in $G$ is a $k$FD-set for some integer $k \geq 1$. Thus a dominating set $S \subseteq V(G)$ is an FD-set in $G$ if for every two distinct vertices $u$ and $v$ from $V(G) \setminus S$, $|N(u) \cap S| = |N(v) \cap S|$. The fair domination number of $G$, denoted $\gamma_{fd}(G)$, is the minimum cardinality of an FD-set. A fair dominating set of cardinality $\gamma_{fd}(G)$ is called a minimum fair dominating set or a $\gamma_{fd}(G)$-set. The concepts of fair domination and $k$-fair domination in graphs were introduced by Caro, Hansberg and Henning [4]. These were used by Bresar and Rall [1] to prove Vizing’s conjecture which states that if $\gamma(G)$ denotes the minimum number of vertices in a dominating set for $G$, then $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$.

For an integer $k \geq 1$, a $k$-fair total dominating set, abbreviated $k$FTD-set, is a total dominating set $S \subseteq V(G)$ such that $|N(u) \cap S| = k$ for every $u \in V(G) \setminus S$. A fair total dominating set (FTD-set) $S$ of a connected graph $G$ is a total dominating set $S$ of $G$ such that for every two distinct vertices $u$ and $v$ of $V(G) \setminus S$, $|N(u) \cap S| = |N(v) \cap S|$; that is, $S$ is both a fair dominating set and a total dominating set of $G$. The fair total domination number of $G$, denoted $\gamma_{ftd}(G)$, is the minimum cardinality of an FTD-set. A fair total dominating set of cardinality $\gamma_{ftd}(G)$ is called a minimum fair total dominating set or a $\gamma_{ftd}$-set of $G$.

2 Preliminary Result

The following observation is worth mentioning.
Remark 2.1 Every FTD-set of a graph $G$ is an FD-set of $G$. In particular, 
$\gamma(G) \leq \gamma_{fd}(G) \leq \gamma_{ftd}(G)$ and $2 \leq \gamma_t(G) \leq \gamma_{ftd}(G)$.

3 Fair Total Domination in the Join of Graphs

Let $A$ and $B$ be sets which are not necessarily disjoint. The disjoint union of $A$ and $B$, denoted by $A \cup B$, is the set obtained by taking the union of $A$ and $B$ treating each element in $A$ as distinct from each element in $B$. The join of two graphs $G$ and $H$ is the graph $G + H$ with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 3.1 [3] Let $G$ and $H$ be connected graphs. Then $C \subseteq V(G + H)$ is a total dominating set in $G + H$ if and only if it satisfies at least one of the following:

(i) $C \cap V(G)$ is a total dominating set in $G$.

(ii) $C \cap V(H)$ is a total dominating set in $H$.

(iii) $C \cap V(G) \neq \emptyset$ and $C \cap V(H) \neq \emptyset$.

Theorem 3.2 [8] Let $G$ and $H$ be connected graphs. Then $S \subseteq V(G + H)$ is an FD-set of $G + H$ if and only if one of the following statements holds:

(a) $S \subseteq V(G)$ and $S$ is a $|S|$FD-set of $G$.

(b) $S \subseteq V(H)$ and $S$ is a $|S|$FD-set of $H$.

(c) $S = V(G) \cup S_H$, where $S_H$ is an $r$FD-set of $H$ for some positive integer $r$.

(d) $S = S_G \cup V(H)$, where $S_G$ is a $k$FD-set of $G$ for some positive integer $k$.

(e) $S = S_G \cup S_H$, where $S_G$ is a $k$FD-set of $G$ and $S_H$ is an $r$FD-set of $H$ for some positive integers $k$ and $r$ such that $k + |S_H| = r + |S_G|$.

Theorem 3.3 Let $G$ and $H$ be connected graphs. Then $S \subseteq V(G + H)$ is an FTD-set of $G + H$ if and only if one of the following statements holds:

(a') $S \subseteq V(G)$ and $S$ is a $|S|$FTD-set of $G$.

(b') $S \subseteq V(H)$ and $S$ is a $|S|$FTD-set of $H$.

(c') $S = V(G) \cup S_H$, where $S_H$ is an $r$FD-set of $H$ for some positive integer $r$. 

(d') \( S = S_G \cup V(H) \), where \( S_G \) is a kFD-set of \( G \) for some positive integer \( k \).

(e') \( S = S_G \cup S_H \), where \( S_G \) is a kFD-set of \( G \) and \( S_H \) is an rFD-set of \( H \) for some positive integers \( k \) and \( r \) such that \( k + |S_H| = r + |S_G| \).

**Proof.** Suppose that \( S \) is an FTD-set of \( G + H \). Then \( S \) is an FTD-set of \( G + H \) by Remark 2.1. Thus, any one of the conditions (a) to (e) of Theorem 3.2 may hold. It remains to show that \( S \) is a total dominating set of \( G \) (resp. \( H \)) if \( S \subseteq V(G) \) (resp. \( S \subseteq V(H) \)). To this end, suppose \( S \subseteq V(G) \). Since \( S \) is a total dominating set of \( G + H \), \( S \cap V(G) \) is a total dominating set in \( G \) by Theorem 3.1(i). Thus, \( S \) is a total dominating set in \( G \). Similarly, if \( S \subseteq V(H) \), then \( S \) is a total dominating set in \( H \) by Theorem 3.1(ii).

Conversely, suppose that (e') holds. By Theorem 3.2, \( S \) is an FTD-set of \( G + H \). Let \( v \in S \). If \( v \in S_G \), then there exists \( w \in S_H \) such that \( vw \in E(G + H) \). If \( v \in S_H \), then there exists \( u \in S_G \) such that \( uv \in E(G + H) \). Therefore, \( S \) is a total dominating set of \( G + H \). Hence, \( S \) is an FTD-set of \( G + H \). If (a'), (b'), (e') or (d') holds, clearly, \( S \) is an FTD-set of \( G + H \). \( \square \)

**Corollary 3.4** Let \( G \) and \( H \) be connected graphs. Then

\[
\gamma_{ftd}(G + H) = \min\{q'_G, q'_H, q_B\},
\]

where

\[
q'_G = \min\{|S| : S \text{ is a } |S|\text{-FTD-set of } G\},
\]

\[
q'_H = \min\{|S| : S \text{ is a } |S|\text{-FTD-set of } H\},
\]

\[
q_B = \min\{2|S_k| - k + r : k, r \in \mathbb{N}, S_k \text{ is a kFD-set of } G \text{ and } H \text{ has an } r\text{-FD-set } S^*_r \text{ with } |S^*_r| = |S_k| - k + r\}.
\]

**Proof.** Let \( S \) be a \( \gamma_{ftd} \)-set of \( G + H \). By Theorem 3.3, \( S \) is of the form given in (a'), (b'), or (e'). Hence, \( \gamma_{ftd}(G + H) = |S| \geq \min\{q'_G, q'_H\} \) if \( S \) is of the form given in (a') or (b'). Suppose \( S = S_G \cup S_H \), where \( S_G \) is a kFD-set of \( G \) and \( S_H \) is an rFD-set of \( H \) for some positive integers \( k \) and \( r \) such that \( r + |S_G| = k + |S_H| \). Then \( \gamma_{ftd}(G + H) = |S| = |S_G| + |S_H| = 2|S_G| + r - k \geq q'_B \). Thus, \( \gamma_{ftd}(G + H) \geq \min\{q'_G, q'_H, q'_B\} \). Next, let \( \rho = \min\{q'_G, q'_H, q'_B\} \). Let \( S_1 \) and \( S_2 \) be a \( |S_1| \)-FTD-set and a \( |S_2| \)-FTD-set of \( G \) and \( H \), respectively, such that \( |S_1| = q'_G \) and \( |S_2| = q'_H \). Let \( S_k \) be a kFD-set of \( G \) and let \( S^*_r \) be an rFD-set of \( H \) for some positive integers \( k \) and \( r \) such that \( |S^*_r| = |S_k| - k + r \) and \( 2|S_k| - k + r = q'_B \). Then \( k + |S^*_r| = r + |S_k| \). Let \( S_3 = S_k \cup S^*_r \). Then \( |S_3| = q'_B \). By Theorem 3.3 (a'), (b'), and (e'), \( S_1, S_2, \) and \( S_3 \) are FTD-sets of \( G + H \). Thus, \( \gamma_{ftd}(G + H) \leq \min\{q'_G, q'_H, q'_B\} \). This proves the desired equality. \( \square \)
4 Fair Total Domination in the Corona of Graphs

Let $G$ and $H$ be graphs of order $m$ and $n$, respectively. The corona of two graphs $G$ and $H$ is the graph $G \circ H$ obtained by taking one copy of $G$ and $m$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex of the $i$th copy of $H$. For every $v \in V(G)$, denote by $H^v$ the copy of $H$ whose vertices are attached one by one to the vertex $v$. Denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\{v\} + H^v$.

**Theorem 4.1** [8, 9] Let $G$ and $H$ be connected graphs. Then $C \subseteq V(G \circ H)$ is an FTD-set in $G \circ H$ if and only if one of the following holds:

(i) $C = V(G)$.

(ii) $C = \bigcup_{v \in V(G)} S_v$, where each $S_v$ is a kFD-set of $H^v$ and $|S_v| = k$.

(iii) $C = V(G) \cup (\bigcup_{v \in V(G)} S_v)$, where $S_v = V(H^v)$ or $S_v$ is a $(k - 1)$FD-set for some $k > 1$ for each $v \in V(G)$.

(iv) $C = A \cup (\bigcup_{v \in V(G)} V(H^v))$, where $A$ is an rFD-set, $C$ is a kFD-set and $|V(H)| = k - r > 0$.

**Theorem 4.2** Let $G$ and $H$ be connected graphs. Then $C \subseteq V(G \circ H)$ is an FTD-set in $G \circ H$ if and only if one of the following holds:

(a) $C = V(G)$.

(b) $C = \bigcup_{v \in V(G)} S_v$, where each $S_v$ is a kFTD-set of $H^v$ and $|S_v| = k$.

(c) $C = V(G) \cup (\bigcup_{v \in V(G)} S_v)$, where $S_v = V(H^v)$ or $S_v$ is a $(k - 1)$FD-set for some $k > 1$ for each $v \in V(G)$.

(d) $C = A \cup (\bigcup_{v \in V(G)} V(H^v))$, where $A$ is an rFD-set, $C$ is a kFD-set and $|V(H)| = k - r > 0$.

**Proof.** Let $C$ be an FTD-set of $G \circ H$. Then, $C$ is an FTD-set of $G \circ H$ by Remark 2.1. Thus, any of (i), (ii), (iii) or (iv) of Theorem 4.1 may hold. If (ii) of Theorem 4.1 holds, then each $S_v$ must be a total dominating set in $H^v$ since $v \notin C$ and $C$ is a total dominating set. Thus, (b) holds.

For the converse, suppose that (a) or (b) holds. Clearly, $C$ is a total dominating set of $G \circ H$. By Theorem 4.1, $C$ is an FTD-set of $G \circ H$. Suppose that (c) or (d) holds. By Theorem 4.1, $C$ is an FTD-set of $G \circ H$. Clearly, $C$ is a total dominating set of $G \circ H$. Therefore, $C$ is an FTD-set of $G \circ H$. □

**Corollary 4.3** Let $G$ be a connected graph and let $H$ be any graph. Then $\gamma_{ftd}(G \circ H) = |V(G)|$.
5 Fair Total Domination in the Composition of Graphs

The composition of two graphs $G$ and $H$ is the graph $G[H]$ with vertex-set $V(G[H]) = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(x,u)(y,v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$.

**Theorem 5.1** [2] Let $G$ and $H$ be connected graphs. Then $C \subseteq V(G[H])$ is a total dominating set in $G[H]$ if and only if $C = \bigcup_{x \in S} \{x\} \times T_x$ and either

(i) $S$ is a total dominating set in $G$, or

(ii) $S$ is a dominating set in $G$ and $T_x$ is a total dominating set in $H$ for all $x \in S \setminus N_G(S)$.

**Lemma 5.2** [8] Let $G$ and $H$ be connected graphs. If $C = \bigcup_{x \in S} \{x\} \times T_x \subseteq V(G[H])$ is an FD-set of $G[H]$, then $S$ is a dominating set of $G$ and $T_x$ is an FD-set of $H$ for each $x \in S$.

**Lemma 5.3** [8] Let $G$ and $H$ be connected graphs. If $C = \bigcup_{x \in S} \{x\} \times T_x \subseteq V(G[H])$ where $S \subseteq V(G)$ is a $|S|$FD-set of $G$, $\langle S \rangle$ is a complete subgraph of $G$, and $T_x$ is a $|T_x|$FD-set of $H$ for each $x \in S$, then $C$ is an FD-set of $G[H]$.

**Theorem 5.4** [8] Let $G$ and $H$ be non-trivial connected graphs. Then $C = V(G) \times T$ where $T \neq V(H)$, is an FD-set of $G[H]$ if and only if $T$ is an FD-set and $G$ is a regular graph.

**Theorem 5.5** [8] Let $G$ and $H$ be non-trivial connected graphs. Then $C = S \times V(H)$ is an FD-set of $G[H]$ if and only if $S$ is an FD-set of $G$.

**Theorem 5.6** Let $G$ and $H$ be non-trivial connected graphs. If $C = \bigcup_{x \in S} \{x\} \times T_x \subseteq V(G[H])$ is an FTD-set of $G[H]$, then $T_x$ is an FD-set of $H$ for each $x \in S$ and either

(a) $S$ is a total dominating set in $G$, or

(b) $S$ is a dominating set in $G$ and $T_x$ is an FTD-set of $H$ for each $x \in S \setminus N_G(S)$.

**Proof.** Since $C$ is an FTD-set of $G[H]$, it is both a total dominating set and FD-set of $G[H]$. By Lemma 5.2, $S$ is a dominating set and $T_x$ is an FD-set of $H$ for each $x \in S$. Moreover, by Theorem 5.1, $S$ is a total dominating set of $G$ or $S$ is a dominating set in $G$ and $T_x$ is a total dominating set in $H$ for all $x \in S \setminus N_G(S)$. This proves the assertion. ☐
Theorem 5.7 Let $G$ and $H$ be connected graphs. If $C = \bigcup_{x \in S} \{x\} \times T_x \subseteq V(G[H])$ where $S \subseteq V(G)$ is a $|S|$FTD-set of $G$, $\langle S \rangle$ is a complete subgraph of $G$, and $T_x$ is a $|T_x|$FTD-set of $H$ for each $x \in S$, then $C$ is an FTD-set of $G[H]$.

Proof. Suppose that $S = \{x\}$. Let $(x, a) \in C$. Since $T_x$ is a total dominating set and $a \in T_x$, $ab \in E(H)$ for some $b \in T_x \setminus \{a\}$. Then $(x, b) \in C$ and $(x, a)(x, b) \in E(G[H])$. Suppose $|S| > 1$. Let $(x, a) \in C$. Then $x \in S$. Since $\langle S \rangle$ is a complete subgraph of $G$, there exists $y \in S$ such that $xy \in E(G)$. Pick any $b \in T_y$. Then $(y, b) \in C$ and $(x, a)(y, b) \in E(G[H])$. In either case, $C$ is a total dominating set of $G[H]$. By Lemma 5.3, $C$ is an FTD-set of $G[H]$. □

Corollary 5.8 Let $H$ be any non-trivial connected graph and $n \geq 2$. Then $\gamma_{ftd}(K_n[H]) \leq 2 \cdot q$, where $q = \min\{|T| : T$ is a $|T|$FTD-set of $H\}$.

Proof. Let $S = \{y, z\} \subseteq V(K_n)$ and let $T$ be a $|T|$FTD-set of $H$ such that $q = |T|$. By Theorem 5.7, $S \times T$ is an FTD-set of $K_n[H]$. Thus, $\gamma_{ftd}(K_n[H]) \leq 2q$. □

Remark 5.9 The strict inequality in Corollary 5.8 can be attained. However, the given upper bound is sharp.

To see this, consider the graphs shown in Figure 1. The shaded vertices in each graph form a $\gamma_{ftd}$-set. Thus, $\gamma_{ftd}(K_4[P_4]) = 4 < 2 \cdot 4 = 2 \cdot q$ and $\gamma_{ftd}(K_4[P_3]) = 2 = 2 \cdot 1 = 2 \cdot q$

![Figure 1: The graphs $K_4[P_4]$ and $K_4[P_3]$](image)

Theorem 5.10 Let $G$ and $H$ be non-trivial connected graphs. Then $C = S \times V(H)$ is an FTD-set of $G[H]$ if and only if $S$ is an FD-set of $G$. Moreover, $C = V(G) \times T$, where $T \neq V(H)$ is an FTD-set of $G[H]$ if and only if $T$ is an FD-set of $H$ and $G$ is a regular graph.

Proof. The results follow from Theorem 5.4, Theorem 5.5, and Theorem 5.1. □
Corollary 5.11 Let $G$ and $H$ be non-trivial connected graphs. Then

$$\gamma_{fd}(G[H]) \leq \gamma_{fd}(G)|V(H)|.$$  

Moreover, if $G$ is regular, then

$$\gamma_{fd}(G[H]) \leq \min\{\gamma_{fd}(G)|V(H)|, \gamma_{fd}(H)|V(G)|\}.$$  

Proof. Let $S$ be a $\gamma_{fd}$-set of $G$. Then $C = S \times V(H)$ is an FTD-set of $G[H]$ by Corollary 5.10. Hence, $\gamma_{fd}(G[H]) \leq |C| = |S||V(H)| = \gamma_{fd}(G)|V(H)|$.

Suppose that $G$ is regular. Let $T$ be a $\gamma_{fd}$-set of $H$. Then $C' = V(G) \times T$ is an FTD-set by Corollary 5.10. Thus, $\gamma_{fd}(G[H]) \leq |C'| = \gamma_{fd}(H)|V(G)|$.

Therefore, $\gamma_{fd}(G[H]) \leq \min\{\gamma_{fd}(G)|V(H)|, \gamma_{fd}(H)|V(G)|\}$. $\square$

Remark 5.12 The strict inequalities in Corollary 5.11 can be attained. However, the given upper bounds are sharp.

To see this, consider the graphs shown in Figure 2. The shaded vertices in each graph form a $\gamma_{fd}$-set. Thus, $\gamma_{fd}(P_3[H]) = 2 < 1 \cdot 4 = \gamma_{fd}(P_3) \cdot |V(H)|$, $\gamma_{fd}(P_3[P_4]) = 4 = 1 \cdot 4 = \gamma_{fd}(P_3) \cdot |V(P_4)|$, $\gamma_{fd}(K_3[P_2]) = 2 = \min\{2, 3\} = \min\{\gamma_{fd}(K_3) \cdot |V(P_2)|, \gamma_{fd}(P_2) \cdot |V(K_3)|\}$, and $\gamma_{fd}(G[C_3]) = 2 < \min\{3, 4\} = \min\{\gamma_{fd}(G) \cdot |V(C_3)|, \gamma_{fd}(C_3) \cdot |V(G)|\}$.

Figure 2: The graphs $P_3[H]$, $P_3[P_4]$, $K_3[P_2]$, and $G[C_3]$.

References


Received: September 21, 2014; Published: November 27, 2014