A Note on Property T for $C^*$-algebras

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Abstract

In this note we introduce an alternative definition of Property T for $C^*$-algebras based on the spectrum of a $C^*$-algebra. We show that a group $G$ has Property T if and only if $C^*_r(G)$ has Property T. In addition, we introduce and investigate relative Property T for $C^*$-algebras.

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1 Introduction

There exist several equivalent statements of Property T for groups. Various authors have tried to extend these definitions from groups to $C^*$-algebras [1, 4, 6]. The original definition of Property T for $C^*$-algebras introduced by Bekka in [1] has deservedly received the most attention [2, 3, 5, 7]. However, a slightly stronger definition of Property T given by Leung and Ng seems to be more
fruitful. In this note we introduce an alternative definition of Property T for $C^*$-algebras using the spectrum of a $C^*$-algebra. Our definition is inspired by a similar definition for groups. In Section 2, we give the definition of Property T and show that a discrete group $G$ has Property T if and only if its reduced group $C^*$-algebra $C^*_r(G)$ has Property T.

In Section 3, we define relative Property T. We concentrate our analysis on Property T relative to the set of finite dimensional Hilbert bimodules.

2 Property T

Let $G$ be a locally compact group and $\hat{G}$ be the set of equivalence classes of irreducible unitary representations of $G$. Then we know that $G$ has Property T if and only if every finite dimensional irreducible representation of $G$ is isolated in $\hat{G}$. Let $A$ be a $C^*$-algebra and $\hat{A}$ be the set equivalence classes of irreducible representations of $A$. Recall that $\hat{A}$ is endowed with the pull-back topology from Prim $(A)$. We introduce the following definition of Property T for $C^*$-algebras.

Definition 1. Let $A$ be a unital $C^*$-algebra. We say that $A$ has Property T if every finite dimensional irreducible representation of $A$ is isolated in $\hat{A}$.

The above definition is similar to the definition proposed by Pavlov and Troitsky in [6]. However, we believe that our definition is more appropriate in at least one important case. In particular, $C(X)$ has Property T if and only if $X$ is finite. This is a natural result as amenability and Property T traditionally only coincide in finite dimensional cases. In general, any finite dimensional $C^*$-algebra has Property T.

Let $G$ be a discrete group. Let $C^*(G)$ denote the group $C^*$-algebra and $C^*_r(G)$ denote the reduced group $C^*$-algebra.

Theorem 2. Let $G$ be a discrete group. Then the following statements are equivalent:

1. $G$ has Property T.
2. $C^*(G)$ has Property T.
3. $C^*_r(G)$ has Property T.

Proof. The equivalence of (1) and (2) is well known. We will show the equivalence of (1) and (3). Suppose that $C^*_r(G)$ does not have Property T. Then there exists a finite dimensional irreducible representation $\pi_0$ of $C^*_r(G)$ and a net $\{\pi_i\}$ in $C^*_r(G)$ such that $\pi_i \to \pi_0$. Since $G$ is embedded in $C^*_r(G)$ we can
take $\pi_0$ and $\{\pi_i\}$ as representations of $G$. We will show that $\pi_0$ is in the closure of the net $\{\pi_i\}$ in $\hat{G}$. Let $s \in G$ be in the intersection $\bigcap_i \ker \pi_i$. Since $s$ is also an element of $C_*^r(G)$, then $s \in \ker \pi_0$. Therefore, $\bigcap_i \ker \pi_i \subseteq \ker \pi_0$. It follows that $G$ does not have Property T.

Conversely, suppose that $G$ does not have Property T. Then the trivial representation of $G$, $1_G$, is not isolated in $\hat{G}$. Let $\{\pi_i\}$ be a net in $\hat{G}$ such that $\pi_i \to 1_G$. Let $\lambda_G$ be the regular representation of $G$. Then $\lambda_G \otimes \pi_i \to \lambda_G$. Since $\lambda_G \otimes \pi_i$ is equivalent to a multiple of $\lambda_G$ we can extend $\lambda_G \otimes \pi_i$ to a representation of $C_*^r(G)$. It follows $\lambda_G \otimes \pi_i \to \lambda_G$ as representations of $C_*^r(G)$. Therefore, $C_*^r(G)$ does not have Property T.

Unfortunately, it remains an open question whether our definition of Property T is equivalent to that of Bekka. We only remark that in the redundant case when a $C^*$-algebra $A$ does not have a tracial state $A$ has Property T by either definition.

3 Relative Property T

In this section we would like to reconsider Bekka’s definition in a more liberal sense. Recall that a Hilbert bimodule over a $C^*$-algebra $A$ is a Hilbert space $H$ carrying a pair of commuting representations, one of $A$ and one of its opposite algebra. A sequence of unit vectors $\{\xi_i\}$ in $H$ is called almost central vectors if $\|a\xi_i - \xi_i a\| \to 0$ for all $a \in A$.

Definition 3. Let $R$ be a set of Hilbert bimodules of $A$. We say that $A$ has Property $(T, R)$ if for every bimodule $H$ in $R$ that has a sequence of almost central vectors there is a nonzero central vector in $H$.

Note that if $R$ is the set of all Hilbert bimodules of $A$, then we obtain Bekka’s original definition of Property T. We are particularly interested in the case when $R$ is the set of all finite dimensional Hilbert bimodules of $A$. We need the following lemma for our main result.

Lemma 4. Let $\pi$ be a finite dimensional representation of $A$ and $\rho$ be an irreducible representation of $A$ such that $\ker \pi \subseteq \ker \rho$. Then $\rho$ is a subrepresentation of $\pi$.

Proof. Since $\pi$ is finite dimensional it decomposes as a finite direct sum of irreducible representations. Then $\ker \pi = \bigcap_{i=1}^n Q_i$, where $Q_i \in \text{Prim}(G)$. Since $\ker \rho$ is prime and $\bigcap Q_i \subseteq \ker \rho$, then $Q_j \subseteq \ker \rho$ for some $j$. But $Q_j$ is the kernel of a finite dimensional irreducible representation of $A$ so $Q_j$ is a maximal ideal. Therefore, $Q_j = \ker \rho$. Note that finite dimensional irreducible representations of a $C^*$-algebra are equivalent if and only if they have the same kernel. It follows that $\rho$ is equivalent to a subrepresentation of $\pi$. \qed
Theorem 5. Let $G$ be a discrete group and let $F$ be the set of finite dimensional Hilbert bimodules of $C^*(G)$. Then $C^*(G)$ has Property $(T,F)$.

Proof. Let $\mathcal{H}$ be a finite dimensional Hilbert bimodule of $C^*(G)$ with a sequence of almost central vectors $\{\xi_i\}$. Define a representation $\pi$ of $G$ on $\mathcal{H}$ by $\pi(s)\xi = s\xi s^{-1}$ for all $s \in G$ and $\xi \in \mathcal{H}$. Then $\pi(s)\xi \to 0$ for all $s \in G$. So the representation $1_G$ is weakly contained in $\pi$. Since $\pi$ is finite dimensional, then by the above lemma $1_G$ is a subrepresentation of $\pi$. Then there exists a nonzero vector $\xi_0 \in \mathcal{H}$ such that $\pi(s)\xi_0 = \xi_0$ for all $s \in G$. Using linearity and continuity we get that $a\xi_0 = \xi_0 a$ for all $C^*(G)$. □

The next result is another example that the restriction to the set of finite dimensional bimodules is generally a weaker condition than the original definition of Property $T$ by Bekka.

Proposition 6. Let $X$ be a compact Hausdorff space and let $F$ be the set of finite dimensional Hilbert bimodules of $C(X)$. Then $C(X)$ has Property $(T,F)$.

Proof. Let $\mathcal{H}$ be a finite dimensional Hilbert bimodule of $C(X)$. Then $\mathcal{H} = L^2(X \times X, \mu)$, where $\mu$ has finite support and

\[(f\xi)(x, y) = f(x)\xi(x, y)\]

\[(\xi f)(x, y) = \xi(x, y)f(y)\]

for all $f \in C(X)$ and $\xi \in L^2(X \times X, \mu)$. Suppose that $\{\xi_i\}$ is a sequence of almost central vectors in $L^2(X \times X, \mu)$. Let $(x, y) \in X \times X$ such that $\mu(x, y) \neq 0$. If $x \neq y$, then there is $g \in C(X)$ such that $g(x) = 1$ and $g(y) = 0$. Since $\mu$ has finite support $|g(x)\xi_i(x, y) - \xi_i(x, y)g(y)| \to 0$. Then $\xi_i(x, y) \to 0$. It follows that there is a point $(x_0, y_0) \in X \times X$ such that $\mu(x_0, x_0) \neq 0$. Then the characteristic function $\xi_0 = \chi_{(x_0,x_0)}$ is a nonzero central vector. □

References


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