Spectral Stability Analysis of a New Difference Scheme of Time Fractional Advection Dispersion Equations

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Abstract

In this paper, a new difference scheme is constructed based on Crank Nicholson difference scheme. It can be used for solving Time Fractional Advection Dispersion Equations involving Caputo fractional derivative. We prove that the proposed method is unconditionally stable by using spectral stability technique. Numerical experiments are presented.

Mathematics Subject Classification: 65N06, 65N12, 65M12

Keywords: Time-Fractional Advection Dispersion Equations, Crank-Nicholson Difference Schemes, Spectral Stability

1 Introduction

Many problems in applied science, physic and engineering are modeled mathematically by the fractional partial differential equations (FPDEs). We can see these models adoption in viscoelasticity [11], finance [10], hydrology [2], engineering [12], control systems [8]. Finite difference methods are very popular method to solve FPDEs [5, 7]. These methods for solving FPDEs
have been studied extensively by many researchers [9, 4, 3, 6, 1]. In this study, we consider time fractional advection dispersion problem
\[ \begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + f(x,t), & (0 < x < 1, 0 < t < 1), \\ u(x,0) = r(x), & 0 \leq x \leq 1, \\ u(0,t) = 0, \quad u(1,t) = 0, & 0 \leq t \leq 1. \end{cases} \] (1)

Here, the term $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ denotes $\alpha$-order Caputo derivative, with the formula:
\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t u_t(x,\tau)(t-\tau)^{-\alpha} d\tau, \quad \text{where} \ 0 < \alpha < 1, \quad (2)
\]
where $\Gamma(\cdot)$ is the Gamma function.

2 Discretization of Problem

We introduce the basic ideas for the numerical solution of the Time Fractional Advection Dispersion Equations by Crank-Nicholson difference scheme.

For some positive integers $M$ and $N$, the grid sizes in space and time for the finite difference algorithm are defined by $h = 1/M$ and $\tau = 1/N$, respectively. The grid points in the space interval $[0, 1]$ are the numbers $x_j = jh$, $j = 0, 1, 2, ..., M$, and the grid points in the time interval $[0, 1]$ are labeled $t_k = k\tau$, $k = 0, 1, 2, ..., N$. The values of the functions $u$ and $f$ at the grid points are denoted $u_j^k = u(x_j, t_k)$ and $f_j^k = f(x_j, t_k)$, respectively. Let $u(x,t)$, $u_t(x,t)$ and $u_{tt}(x,t)$ are continuous on $[0, 1]$.

As in the classical Crank-Nicholson difference scheme, a discrete approximation to the fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ at $(x_j, t_{k+\frac{1}{2}})$ can be obtained by the following quadrature formula [6]:
\[
\frac{\partial^\alpha u(x_j, t_{k+\frac{1}{2}})}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{k+1/2}} u_t(x,s)(t_{k+1/2} - s)^{-\alpha} ds \\
= \frac{1}{\Gamma(1-\alpha)} \left[ \int_0^{t_k} u_t(x,s) \left( (k + \frac{1}{2})\tau - s \right)^{-\alpha} ds \right. \\
+ \left. \int_{t_k}^{t_{k+1/2}} \frac{u_{j+1}^{k+1} - u_j^k}{\tau} + O(\tau) \left( (k + \frac{1}{2})\tau - s \right)^{-\alpha} ds \right]
\]
So, we obtain,
\[
\frac{\partial^\alpha u(x_j, t_{k+\frac{1}{2}})}{\partial t^\alpha} = \frac{1}{\Gamma(1 - \alpha)} \frac{1}{1 - \alpha} \left\{ \frac{(u_{j+1}^k - u_j^k)}{2^{1-\alpha}} \right\}
+ \sum_{m=1}^{k} [u_j^m - u_j^{m-1}] \left\{ (k - m + \frac{3}{2})^{1-\alpha} - (k - m + \frac{1}{2})^{1-\alpha} \right\}
+ R_1 + R_2.
\]

where
\[
R_1 = \frac{1}{\Gamma(1 - \alpha)} \sum_{m=1}^{k} \left[ \int_{(m-1)\tau}^{m\tau} (s - t_{m-\frac{1}{2}}) u_{tt}(x_j, c_m) \left( \frac{k + 1}{2}\tau - s \right)^{-\alpha} ds \right]
\]

and
\[
R_2 = \frac{1}{\Gamma(2 - \alpha)} \frac{1}{2^{1-\alpha}} O(\tau^{2-\alpha}).
\]

Here,
\[
R_1 = \frac{1}{\Gamma(1 - \alpha)} \sum_{m=1}^{k} \left[ \int_{(m-1)\tau}^{m\tau} (s - t_{m-\frac{1}{2}}) u_{tt}(x_j, c_m) \left( \frac{k + 1}{2}\tau - s \right)^{-\alpha} ds \right]
\]

\[
\leq \frac{\max_{1 \leq j \leq n} |u_{tt}(x_j, c_m)|}{\Gamma(1 - \alpha)} \tau^{2-\alpha}.
\]

Setting \( \sigma = \frac{1}{\Gamma(2 - \alpha)} \frac{1}{\tau^{1-\alpha}} \) and \( w_j = \sigma ((j + 1/2)^{1-\alpha} - (j - 1/2)^{1-\alpha}) \), finally we have obtained the following approximation;
\[
\frac{\partial^\alpha u(x_j, t_{k+\frac{1}{2}})}{\partial t^\alpha} = w_1 u^k + \sum_{m=1}^{k-1} (w_{k-m+1} - w_{k-m}) u^m - w_k u^0 + \sigma \frac{(u_{j+1}^k - u_j^k)}{2^{1-\alpha}}
+ O(\tau^{2-\alpha}).
\] (3)

In addition for \( k = 0 \) there is no these terms \( w_1 u^k \) and \( w_k u^0 \)

On the other hand, we have
\[
\frac{\partial^2 u(x_j, t_{k+\frac{1}{2}})}{\partial x^2} = \frac{1}{2} \left[ \frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{h^2} + \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h^2} \right] + O(h^2). \] (4)
3 The Proposed Difference Scheme

Using the approximations at (3) and (4), we obtain the following difference scheme which is accurate of order $O(\tau^{2-\alpha} + h^2)$:

\[
\begin{cases}
    w_1 u_j^k + \sum_{m=1}^{k-1} (w_{k-m+1} - w_{k-m}) u_j^m - w_k u_j^0 + \sigma \frac{(u_{k+1}^k - u_k^k)}{2h^2} \\
    - \left[ \frac{u_{j+1}^{k+1} - 2u_{j+1}^k + u_{j+1}^{k-1}}{2h^2} \right] + \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{2h^2} \right] + \left[ \frac{u_{j+1}^{k+1} - u_{j-1}^{k+1}}{4h} \right] \\
    = f(x_j, t_k + \frac{\tau}{2}), \quad 0 \leq k \leq N - 1, \ 1 \leq j \leq M - 1,

\end{cases}
\]

(5)

We can arrange the system above, to obtain

\[
\begin{align*}
    U_j^0 &= r(x_j), \quad 1 \leq j \leq M, \\
    U_0^k &= 0, \quad U_M^k = 0, \quad 0 \leq k \leq N.
\end{align*}
\]

The difference scheme above can be written in matrix form:

\[
DU_{j+1} + EU_j + FU_{j-1} = \varphi_j \text{ where } \varphi_j = \left[ \varphi_j^0, \varphi_j^1, \varphi_j^2, ..., \varphi_j^N \right]^T, \varphi_j^0 = r(x_j), \varphi_j^k = f(x_j, t_{k+\frac{T}{2}}), \ 1 \leq j \leq M, 1 \leq k \leq N, \text{ and } U_j = \left[ U_j^0, U_j^1, U_j^2, ..., U_j^N \right]^T.
\]

Here $D_{(N+1) \times (N+1)}$ and $E_{(N+1) \times (N+1)}$ are the matrices of the form

\[
D = \left( \begin{array}{cccc}
    0 & 1 & & \\
    1 & 1 & & \\
    & & \ddots & \\
    & & & 1 & 1
\end{array} \right)
\]

\[
E = \left[ \begin{array}{cccc}
    1 & a \\
    b & b + w_1 & & \\
    -w_1 & b + w_1 & a \\
    -w_2 & w_2 - w_1 & b + w_1 & a \\
    & \ddots & \ddots & \ddots \\
    & & w_{N-1} & w_{N-2} & \ddots & w_2 - w_1 & b + w_1 & a
\end{array} \right]
\]
Spectral stability

\[ F = \left( -\frac{1}{2h^2} - \frac{1}{4h} \right) \begin{bmatrix} 0 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & 1 & 1 \end{bmatrix} \]

where \( a = \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} \), \( b = -\frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} \).

Using the idea on the modified Gauss-Elimination method, we can convert into the following form:

\[ U_j = \psi_{j+1}U_{j+1} + \mu_{j+1}; j = M, \ldots, 2, 1, 0. \]

Then, we write

\[ D + E\psi_{j+1} + F\psi_j\psi_{j+1} = 0, \]
\[ E\mu_{j+1} + F\psi_j\mu_{j+1} + F\mu_j = \varphi_j, \]

where \( 1 \leq j \leq M \).

So, we obtain the following pair of formulas:

\[ \psi_{j+1} = -(E + F\psi_j)^{-1}D, \mu_{j+1} = (E + F\psi_j)^{-1}(\varphi_j - F\mu_j), \]

where \( 1 \leq j \leq M \).

We will prove that \( \rho \left( \psi_j \right) < 1, 1 \leq j \leq M \), by induction. Since \( \psi_1 \) is a zero matrix \( \rho \left( \psi_1 \right) = 0 < 1 \). Moreover, \( \psi_2 = -E^{-1}D \),

\[ \rho(\psi_2) = \rho(-E^{-1}D) = \frac{-1}{\frac{\sigma}{2^{1-\alpha}} + \frac{1}{4h^2}} \cdot \left( -\frac{1}{2h^2} + \frac{1}{4h} \right) = \frac{\frac{\sigma}{2^{1-\alpha}} + \frac{1}{4h^2}}{M^2 + \frac{1}{2^{1-\alpha}}} \]

since \( \psi_2 \) is of the form

\[ \psi_2 = \begin{bmatrix} 0 & -\frac{1}{2h^2} + \frac{1}{4h} & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & \left( \frac{\sigma}{2^{1-\alpha}} + \frac{1}{4h^2} \right) & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & \left( \frac{\sigma}{2^{1-\alpha}} + \frac{1}{4h^2} \right) & \cdots & \cdots & \cdots & \cdots \\ * & * & * & \left( \frac{\sigma}{2^{1-\alpha}} + \frac{1}{4h^2} \right) & \cdots & \cdots & \cdots \\ * & * & * & * & \cdots & \cdots & \cdots \\ * & * & * & * & * & \cdots & \cdots \\ * & * & * & * & * & * & \left( M+1 \right)(M+1) \end{bmatrix} \]

\[ \sigma = \frac{1}{\Gamma(2-\alpha)} \frac{1}{\tau^\alpha} > 0, \text{ therefore, } \rho \left( \psi_2 \right) < 1. \]

Now, assume \( \rho \left( \psi_j \right) < 1 \). We find that

\[ \psi_{j+1} = -(E + F\psi_j)^{-1}D \]

\[ = \left( \frac{1}{2h^2} - \frac{1}{4h} \right) \begin{bmatrix} 0 & E_{2,2} - \left( \frac{1}{2h^2} + \frac{1}{4h} \right)\psi_{j,2,2} & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & \left( \frac{\sigma}{2^{1-\alpha}} + \frac{1}{4h^2} \right) & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & \left( \frac{\sigma}{2^{1-\alpha}} + \frac{1}{4h^2} \right) & \cdots & \cdots & \cdots & \cdots \\ * & * & * & \left( \frac{\sigma}{2^{1-\alpha}} + \frac{1}{4h^2} \right) & \cdots & \cdots & \cdots \\ * & * & * & * & \cdots & \cdots & \cdots \\ * & * & * & * & * & \cdots \cdots & \cdots \\ * & * & * & * & * & * & \left( M+1 \right)(M+1) \end{bmatrix} \]
and we already know that $E_{j,j} = \frac{\sigma^2}{2^{\alpha-1}} + \frac{1}{h^2}$ and $\psi_{j,r,r} = \rho(\psi_{j})$ for $2 \leq r \leq M+1$:

$$
\rho(\psi_{j+1}) = \frac{\frac{1}{2h^2} - \frac{1}{4h} + \frac{1}{M} \rho(\psi_{j})}{\frac{\sigma^2}{2^{\alpha-1}} + \frac{1}{h^2} - \left(\frac{1}{2h^2} + \frac{1}{4h}\right) \rho(\psi_{j})} = \frac{M^2 - \frac{M^2}{2} + \frac{M}{4}}{M^2 - (\frac{M^2}{2} + \frac{M}{4}) \rho(\psi_{j}) + \frac{\sigma^2}{2^{\alpha-1}}}
$$

Since $0 \leq \rho(\psi_{j}) < 1$, it follows that $\rho(\psi_{j+1}) < 1$. So, $\rho(\psi_{j}) < 1$ for any $j$, where $1 \leq j \leq M$.

### 4 Numerical Example

Consider this problem,

$$
\begin{align*}
\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} &= \frac{\partial^2 u(t,x)}{\partial x^2} - \frac{\partial u(t,x)}{\partial x} + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} (1-x) \sin(x) + t^2 \left[(3-x) \cos(x) - x \sin(x)\right], \\
(0 < x < 1, 0 < t < 1), \\
u(0, x) &= 0, \ 0 \leq x \leq 1, \\
u(t, 0) &= 0, \ u(t, 1) = 0, \ 0 \leq t \leq 1.
\end{align*}
$$

Exact solution of this problem is $u(t, x) = t^2 (1-x) \sin(x)$. The errors for some $M$ and $N$ are given in Figure 1. The spectral radius of the each iteration matrix, i.e., $\rho(\psi_{j+1})$, used to solve the problem is calculated and listed in the Table 1. The errors when solving this problem are listed in the Table 2 for various values of time and space nodes. The errors in the Table 2 are calculated by the formula $\max_{0 \leq k \leq N, 0 \leq n \leq M} | u(t_k, x_n) - U^k_n |$.

![Figure 1: The errors when t=1 for some M and N](image-url)
Table 1: The spectral radius of iteration matrices for some values of M and N

<table>
<thead>
<tr>
<th>Number of iteration matrix</th>
<th>Spectral Radius</th>
<th>Spectral Radius</th>
<th>Spectral Radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho(\psi_1) )</td>
<td>0.4802411174</td>
<td>0.4793056884</td>
<td>0.4790508638</td>
</tr>
<tr>
<td>( \rho(\psi_2) )</td>
<td>0.6364947997</td>
<td>0.6344690943</td>
<td>0.6337513945</td>
</tr>
<tr>
<td>( \rho(\psi_3) )</td>
<td>0.7118649113</td>
<td>0.7085535371</td>
<td>0.7075603248</td>
</tr>
<tr>
<td>( \rho(\psi_4) )</td>
<td>0.7550044405</td>
<td>0.7505134082</td>
<td>0.7490016578</td>
</tr>
<tr>
<td>( \rho(\psi_5) )</td>
<td>0.7820049654</td>
<td>0.7763482968</td>
<td>0.7754537826</td>
</tr>
<tr>
<td>( \rho(\psi_6) )</td>
<td>0.8001067912</td>
<td>0.7933975336</td>
<td>0.7924815588</td>
</tr>
<tr>
<td>( \rho(\psi_7) )</td>
<td>0.8124988260</td>
<td>0.8052532762</td>
<td>0.8025564446</td>
</tr>
<tr>
<td>( \rho(\psi_8) )</td>
<td>0.8214725395</td>
<td>0.8141886884</td>
<td>0.8110776773</td>
</tr>
<tr>
<td>( \rho(\psi_9) )</td>
<td>0.8277824972</td>
<td>0.8197635954</td>
<td>0.8169194066</td>
</tr>
<tr>
<td>( \rho(\psi_{10}) )</td>
<td>0.8325631637</td>
<td>0.8238214952</td>
<td>0.820654815</td>
</tr>
<tr>
<td>( \rho(\psi_{11}) )</td>
<td>0.8359543147</td>
<td>0.8267723166</td>
<td>0.8230767027</td>
</tr>
<tr>
<td>( \rho(\psi_{12}) )</td>
<td>0.8387017227</td>
<td>0.8290432667</td>
<td>0.8248185247</td>
</tr>
<tr>
<td>( \rho(\psi_{13}) )</td>
<td>0.8406052926</td>
<td>0.8293705993</td>
<td>0.8277380192</td>
</tr>
<tr>
<td>( \rho(\psi_{14}) )</td>
<td>0.8421457078</td>
<td>0.8308987827</td>
<td>0.8272842853</td>
</tr>
<tr>
<td>( \rho(\psi_{15}) )</td>
<td>0.8431242008</td>
<td>0.8317760705</td>
<td>0.8290617832</td>
</tr>
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</table>

Table 2: The errors for some values of M, N and \( \alpha \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( M )</th>
<th>( \alpha = 0.3 )</th>
<th>( \alpha = 0.5 )</th>
<th>( \alpha = 0.8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Error(( \alpha, \tau ))</td>
<td>Err. rate</td>
<td>Error(( \alpha, \tau ))</td>
</tr>
<tr>
<td>8</td>
<td>16</td>
<td>0.001804</td>
<td>-</td>
<td>0.0019673</td>
</tr>
<tr>
<td>16</td>
<td>16</td>
<td>0.000443</td>
<td>4.02</td>
<td>0.000405</td>
</tr>
<tr>
<td>32</td>
<td>16</td>
<td>0.000111</td>
<td>4.03</td>
<td>0.000098</td>
</tr>
</tbody>
</table>

5 Conclusion

In this work, \( O(\tau^{2-\alpha} + h^2) \) order approximation for the Caputo fractional derivative based on the Crank-Nicholson difference scheme was successfully applied to solve the time-fractional Advection Dispersion equations. It is proven that the time-fractional Crank-Nicholson difference scheme is unconditionally stable by spectral stability analysis. Numerical results are in good agreement with the theoretical results.
References


Spectral stability

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