On Convergence of the Interval Zoro Symmetric
Single Step Procedure for Polynomial Zeros

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Abstract

This paper describes the convergence analysis of the procedure called the interval zoro symmetric single-step procedure IZSS2-5D which we have earlier proposed. The analysis performed shows that the rate of convergence of this procedure is at least eight.

Keywords: Interval iterative, interval zoro, simultaneous inclusion

1. Introduction

Iterative procedures for simultaneous inclusion of simple polynomial zeros were discussed by Monsi and Wolfe [1], Jamaludin et al. [2-6], Monsi et al. [7,8], Sham et al. [9-11] and Bakar et al. [12]. Our interest lies in the procedure proposed by Jamaludin et al. [2] as in Section 2, which was shown to be convergent numerically in terms of shorter CPU times and lesser number of iterations using five test polynomials with $w^{(i)} \leq 10^{-10}$ as the stopping criterion.
2. The interval zero symmetric single-step procedure IZSS2-5D

Consider \( p : R^1 \to R^1 \) a polynomial of degree \( n > 1 \) defined by

\[
p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_0 = \prod_{j=1}^{n} (x - x_j^*)
\]

where \( a_i \in R^1 (i = 1, ..., n) \) are given. Suppose that \( x_j^* \in R (i = 1, ..., n) \), \( p \) has \( n \) distinct values and that \( X_j^{(0)} \in I(R) \) (set of real intervals) \( (i = 1, ..., n) \) are such that

\[
X_j^{(0)} (i = 1, ..., n)
\]

and \( X_j^{(0)} \cap X_j^{(0)} = \emptyset, (i, j = 1, ..., n; i \neq j) \)

For \( k \geq 0 \)

\[
X_j^{(k)} (i = 1, 2, ..., n) \quad \text{(Initial intervals)} \tag{1a}
\]

\[
x_j^{(k)} = \text{midpoint} \left( X_j^{(k)} \right), (i = 1, ..., n) \tag{1b}
\]

\[
\delta_j^{(k)} = -\frac{p(x_j^{(k)})}{p(x_j^{(0)})} \quad (i = 1, ..., n) \tag{1c}
\]

\[
X_j^{(k,1)} = \left\{ x_j^{(k)} + \frac{\delta_j^{(k)}}{1 + \delta_j^{(k)} \left( \sum_{j=1}^{k} \frac{1}{x_j^{(k)} - x_j^{(k+1)}} + \sum_{j=k+1}^{n} \frac{1}{x_j^{(k)} - x_j^{(k+1)}} \sum_{i=k+1}^{n} 1 \right)} \right\} \cap X_j^{(k)} \quad (i = 1, ..., n) \tag{1d}
\]

\[
X_j^{(k,2)} = \left\{ x_j^{(k)} + \frac{\delta_j^{(k)}}{1 + \delta_j^{(k)} \left( \sum_{j=1}^{k} \frac{1}{x_j^{(k)} - x_j^{(k+1)}} + \sum_{j=k+1}^{n} \frac{1}{x_j^{(k)} - x_j^{(k+1)}} \sum_{i=k+1}^{n} 1 \right)} \right\} \cap X_j^{(k,1)} \quad (i = n, ..., 1) \tag{1e}
\]

\[
X_j^{(k,3)} = \left\{ x_j^{(k)} + \frac{\delta_j^{(k)}}{1 + \delta_j^{(k)} \left( \sum_{j=1}^{k} \frac{1}{x_j^{(k)} - x_j^{(k+1)}} + \sum_{j=k+1}^{n} \frac{1}{x_j^{(k)} - x_j^{(k+1)}} \sum_{i=k+1}^{n} 1 \right)} \right\} \cap X_j^{(k,2)} \quad (i = 1, ..., n) \tag{1f}
\]

\[
X_j^{(k,4)} = X_j^{(k,3)} \quad (i = 1, ..., n) \tag{1g}
\]
Steps (1a) – (1e) were developed by Jamaludin et al. [5], where the rate of convergence was at least six.

3. The rate of convergence of IZSS2-5D

Now we have some descriptions of the Algorithm IZSS2-5D regarding the conditions, inclusion, convergent and the rate of convergence.

**Theorem 1:** Let \( p \) be defined by \( p(x) = \sum_{i=0}^{n} a_i x^i \), where \( a_i \neq 0 \). If (i) \( p \) has \( n \) distinct zeros \( x_i(i=1,\ldots,n) \), \( x_i \in X_i^{(0)} \) and \( X_i^{(0)} \cap X_j^{(0)} = \emptyset \), \( (i, j=1,\ldots,n; i \neq j) \) hold; (ii) \( 0 \not\in D_i \in I(R) \) \( (D_i=[d_{ij},d_{ij}]) \) is such that \( p(x) \in D_i \) \( (\forall x \in X_i^{(0)}) \), \( (i=1,\ldots,n) \) and \( w(x_i^{(k+1)}) \leq \frac{1}{2} \left( 1 - \frac{d_{ij}}{d_{ij}} \right) w(x_i^{(k)}), \) holds (where \( w(X_i^{(k)}) \leq \sum \left[ \frac{x_i^{(k)}, x_j^{(k)}}{x_i^{(k)} - x_j^{(k)}} \right] = \alpha^{(k)} - \beta^{(k)} \)); (iii) the sequence \( \{X_i^{(k)}\} \) \( (i=1,\ldots,n) \) are generated from (1) , then (iv) \( \forall k \geq 0 \) \( x_i^{*} \in X_i^{(k,e)} \subseteq X_i^{(k)} \) \( (i=1,\ldots,n) \); (vi) \( X^{(0,1)} \supseteq X^{(0,2)} \supseteq \ldots \) with \( \lim_{k \to \infty} X^{(k)} = x_i, X_i^{(k)} \to x_i (k \to \infty) (i=1,\ldots,n) \), and \( \text{O}_{n}(IZSS2-5D.x_i^{*}) \geq 8 \) \( (i=1,\ldots,n) \).

**Proof**

The proofs of (iv) and (v) are identical to the proofs in Monsi and Wolfe [1]. Now the proof of (vi) is as follows.

By (1d), (1e) and (1f), \( \exists \alpha > 0 \) such that \( \forall k \geq 0 \),

\[
    w_i^{(k,1)} \leq \beta \left( w_i^{(k,0)} \right)^2 \left\{ \sum_{j=1}^{i-1} w_j^{(k,1)} + \sum_{j=i+1}^{n} \left( w_j^{(k,0)} \right)^2 \right\} \quad (i=1,\ldots,n),
\]

and

\[
    w_i^{(k,2)} \leq \beta \left( w_i^{(k,1)} \right)^2 \left\{ \sum_{j=1}^{i-1} w_j^{(k,2)} + \sum_{j=i+1}^{n} \left( w_j^{(k,1)} \right)^2 \right\} \quad (i=2,\ldots,n),
\]

and

\[
    w_i^{(k,3)} \leq \beta \left( w_i^{(k,2)} \right)^2 \left\{ \sum_{j=1}^{i-1} w_j^{(k,3)} + \sum_{j=i+1}^{n} \left( w_j^{(k,2)} \right)^2 \right\} \quad (i=3,\ldots,n),
\]

where

\[
    w_i^{(k,s)} = (n-1) \alpha \beta w \left( X_i^{(k,x)} \right) \quad (s=0,1,2),
\]

and

\[
    \beta = \frac{1}{n-1}
\]
Let
\[ u_i^{(1,1)} = \begin{cases} 4 & (i = 1, \ldots, n-1) \\ 6 & (i = n) \end{cases} \quad (7) \]
\[ u_i^{(1,2)} = \begin{cases} 8 & (i = 1) \\ 6 & (i = 2, \ldots, n) \end{cases} \quad (8) \]
\[ u_i^{(1,3)} = \begin{cases} 8 & (i = 1, \ldots, n-1) \\ 10 & (i = n) \end{cases} \quad (9) \]

and for \((r = 1, 2)\), let
\[ u_i^{(k+1,r)} = \begin{cases} 8u_i^{(k,r)} & (i = 1, \ldots, n-1) \\ 8u_i^{(k,r)} + 2 & (i = n) \end{cases}, \quad (10) \]

Then by (7) - (10), for \((\forall k \geq 0)\)

\[ u_i^{(k,1)} = \begin{cases} 4(8^{(k-1)}) & (i = 1, \ldots, n-1) \\ \frac{44}{7}(8^{(k-1)}) - \frac{2}{7} & (i = n) \end{cases} \quad (11) \]

\[ u_i^{(k,2)} = \begin{cases} 8(8^{(k-1)}) & (i = 1) \\ 6(8^{(k-1)}) & (i = 2, \ldots, n-1) \\ \frac{44}{7}(8^{(k-1)}) - \frac{2}{7} & (i = n) \end{cases} \quad (12) \]

\[ u_i^{(k,3)} \geq \begin{cases} 8 \left( \frac{8^{(k-1)}}{2} \right) & i \neq 1 \text{ or } \ldots, \\ \frac{72}{7} (8^{(k-1)}) - \frac{2}{7} & (i = n) \end{cases} \quad (13) \]

Without any loss of generality, suppose that \(w_i^{(0,0)} \leq h < 1 \quad i \in 1, \ldots, n \quad (14)\)

Then by inductive argument it follows from (2) - (14) that for \((i = 1, \ldots, n)\) \((k \geq 0)\)

\[ w_i^{(k,1)} \leq h^{w_i^{(k+1,1)}}, \quad w_i^{(k,2)} \leq h^{w_i^{(k+1,2)}}, \quad \text{and} \quad w_i^{(k,3)} \leq h^{w_i^{(k+1,3)}}, \]

whence by (11) and (14), \(w_i^{(k+1)} \leq h^{w_i^{(k)}} \quad (i \neq 1, \ldots, n) \quad (15)\)
On convergence of the interval zoro symmetric single step procedure

So, \(\forall k \geq 0\), by (5) - (15), \(w(X_i^{(k)}) \leq \left(\frac{\beta}{\alpha}\right)^{h^{k}}\) \((i = 1,\ldots,n)\), \(\alpha > 0\). \hspace{1cm} (16)

Let \(w^{(k)} = \max_{i \in I} \left\{w\left(X_i^{(k)}\right)\right\}\) \hspace{1cm} (17)

Then by (16) and (17) \(w^{(k)} \leq \left(\frac{\beta}{\alpha}\right)h^{k}\) \((\forall k \geq 0)\).

So, by definition of R-factor in Monsi and Wolfe [1], we have

\[ R_s(w^{(k)}) = \limsup_{k \to \infty} \left\{ \left(\frac{\beta}{\alpha}\right)^{h^{k}} \right\} = \lim_{k \to \infty} \left\{ \left(\frac{\beta}{\alpha}\right)^{h^{k}} \right\} = h < 1. \]

Therefore, it is proven that \(O_R(\text{IZSS2-5D}, x^*) \geq 8\) \((i = 1,\ldots,n)\). ■

4. Conclusion

It has been shown analytically in Section 3 that IZSS2-5D has a faster rate of convergence of at least eight thus enhances the rate of convergence.

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References


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