On Positive Stable Realization for Continuous Linear Singular Systems

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Abstract

In this paper, a study on the positive stable realization problem for continuous linear singular systems is established. A necessary and sufficient condition for existence of a positive stable realization of an improper transfer matrix is proposed. It is showed that the existence of a positive stable realization of an improper transfer matrix is equivalent to existence of positive stable realization of a strictly proper transfer matrix which corresponding to it.

Mathematics Subject Classification: 93C05, 93D15, 93D20

Keywords: Positive stable realization, singular system, transfer matrix

1 Introduction

The realization of a transfer function is a classical problem in system theory. Such a problem has a complete solution in the case of standard linear systems. However, when constraints on the realization, i.e. on positivity, are imposed, the problem becomes much more complicated. An overview on the positive realization problem for standard linear systems is given in [1], [3]. Currently, Kaczorek [5] propose a procedure to compute the positive stable realization for the case of discrete time, whereas the stability of positive standard linear systems is discussed in [7] for the case of continuous time.
The positive stable realization problem of the more general linear systems, i.e. singular systems, is also already discussed in [6] for the case of discrete time and in [8] for the case of continuous time without a stability condition. Unfortunately, the positive stable realization for a transfer matrix need not always exist. Therefore, the criteria for existence a positive stable realization of a transfer matrix becomes important to be established. In this paper, we discuss a necessary and sufficient condition for existence of positive stable realization for continuous linear singular system.

The following notation will be used:

- $\mathbb{R}$ - the set of real numbers,
- $\mathbb{R}^{n \times m}$ - the set of $n \times m$ real matrices,
- $\mathbb{R}^{n \times m}_+$ - the set of $n \times m$ real matrices with nonnegative entries,
- $\mathbb{R}^{n \times m}_-$ - the set of $n \times m$ real matrices with nonpositive entries,
- $\mathbb{R}^{n \times 1}$ - the set of $n \times 1$ real matrices,
- $\mathcal{M}^n$ - the set of $n \times n$ Metzler matrices,
- $I_n$ - the $n \times n$ identity matrix and
- $O$ - the zero matrix.

### 2 Background Material and Problem Statement

Consider the singular continuous linear system

$$
\dot{w}(t) = Kw(t) + Lu(t), \quad w(0) = w_0
$$

where $w(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $v(t) \in \mathbb{R}^p$ denote state, control and output vector, respectively, and $E, K \in \mathbb{R}^{n \times n}, L \in \mathbb{R}^{n \times m}, M \in \mathbb{R}^{p \times n}$. If $E$ is nonsingular, then the system (1) is reduced in to a standard system

$$
\dot{w}(t) = Aw(t) + Bu(t), \quad w(0) = w_0
$$

where $A = E^{-1}K$ and $B = E^{-1}L$.

In what follows we give some useful definitions and theorems.

**Definition 1** [9] *The standard system (2) is called positive if for any initial state $w(0) = w_0 \in \mathbb{R}^n$ and all control $u(t) \in \mathbb{R}^m$ with $t \geq 0$, then $w(t) \in \mathbb{R}^n_+$ and $v(t) \in \mathbb{R}^p_+$. Furthermore, it is called stable if $\lim_{t \to \infty} w(t) = 0$ for all $w_0 \in \mathbb{R}^n_+$.***

The standard system (2) is called stable positive if it is positive and stable. The criterion for positive stable of the standard system (2) has been reported by some authors, see [1], [4], [9].

**Theorem 1** [4, 9] *The standard system (2) is positive stable if and only if

$$
A \in \mathcal{M}^{ns}, \quad B \in \mathbb{R}^{n \times m}_+, \quad \text{and} \quad M \in \mathbb{R}^{p \times n}_+,
$$

where $\mathcal{M}^{ns}$ denotes the set of $n \times n$ stable Metzler matrices.*
The transfer matrix of the standard system (2) is given by
\[ G(s) = M (sI_n - A)^{-1}B \in \mathbb{R}^{p \times m}(s). \] (4)

The transfer matrix \( G(s) \) is called proper if and only if
\[ \lim_{s \to \infty} G(s) = K \in \mathbb{R}^{p \times m}, \] (5)
and it is called strictly proper if \( K = 0 \). Otherwise, the transfer matrix is called improper.

**Definition 2** [4] Matrices \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, M \in \mathbb{R}^{p \times n} \) are called a positive stable realization of transfer matrix \( G(s) \) if they satisfy (4) and the system standard (2) is stable positive.

In [4] is stated that the matrices \( A, B, M \) are a positive stable realization of the transfer matrix \( G(s) \) if and only if (3) holds.

It is now assumed that the matrix \( E \) is singular and there exists a complex number \( s \) such that
\[ \det (sE - K) \neq 0. \] (6)

**Definition 3** [9] The singular system is called positive if for any admissible initial state \( w(0) = w_0 \in \mathbb{R}^n_+ \) and all control \( u(t) \in \mathbb{R}^m_+ \) such that \( u^{(j)}(t) = \frac{d^j u(t)}{dt^j} \in \mathbb{R}_+^m \) with \( t \geq 0, j = 1, 2, \ldots, q \), then \( w(t) \in \mathbb{R}^n_+ \) and \( v(t) \in \mathbb{R}^p_+ \). It is called stable if \( \lim_{t \to \infty} w(t) = 0 \) for all admissible \( w_0 \in \mathbb{R}_+^n \).

The transfer matrix of the descriptor system (1) is given by
\[ T(s) = M (sE - K)^{-1} L \in \mathbb{R}^{p \times m}(s). \] (7)

**Definition 4** The matrices \( E, K \in \mathbb{R}^{n \times n}, L \in \mathbb{R}^{n \times m}, M \in \mathbb{R}^{p \times n} \) are called a positive realization of transfer matrix \( T(s) \) if they satisfy (7) and the singular system (1) is positive. Moreover, these matrices \( E, K, L, M \) are called a positive stable realization of transfer matrix \( T(s) \) if they are a positive realization and the singular system (1) is stable.

If the nilpotency index \( q \) of the matrix \( E \) is greater than or equal to 1, then the transfer matrix (7) is improper and can be always written as the sum of strictly proper transfer matrix \( T_{sp}(s) \) and the polynomial transfer matrix \( T_{pl}(s) \) [6], namely,
\[ T(s) = T_{sp}(s) + T_{pl}(s). \] (8)
It is well known that under the transformation
\[ w(t) = P \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \]
for a nonsingular matrix \( P \in \mathbb{R}^{n \times n} \), with \( y_1(t) \in \mathbb{R}^n \), \( y_2(t) \in \mathbb{R}^{n-n_1} \) and the assumption (6), the system (1) can be always decomposed into the following system [2]:
\[
\begin{bmatrix}
I_{n_1} & O \\
O & N
\end{bmatrix}
\begin{bmatrix}
\dot{y}_1(t) \\
\dot{y}_2(t)
\end{bmatrix} =
\begin{bmatrix}
K_1 & O \\
O & I_{n-n_1}
\end{bmatrix}
\begin{bmatrix}
y_1(t) \\
y_2(t)
\end{bmatrix} +
\begin{bmatrix}
L_1 \\
L_2
\end{bmatrix} u(t)
\]
\v(t) =
\begin{bmatrix}
M_1 & M_2
\end{bmatrix}
\begin{bmatrix}
y_1(t) \\
y_2(t)
\end{bmatrix}
\]
where \( K_1 \in \mathbb{R}^{n_1 \times n_1} \), \( L_1 \in \mathbb{R}^{n_1 \times m} \), \( L_2 \in \mathbb{R}^{(n-n_1) \times m} \), \( M_1 \in \mathbb{R}^{p \times n_1} \), \( M_2 \in \mathbb{R}^{p \times (n-n_1)} \), and \( N \) is a nilpotent matrix.

The positive stable realization problem for the singular system can be stated as follows. Given an improper transfer matrix
\( T(s) \in \mathbb{R}^{p \times m}(s) \), find the matrices \( \tilde{E}, \tilde{K} \in \mathbb{R}^{n \times n} \), \( \tilde{L} \in \mathbb{R}^{n \times m} \), \( \tilde{M} \in \mathbb{R}^{p \times n} \), with \( \det(\tilde{E}) = 0 \), such that
\[ T(s) = \tilde{M} \left( s \tilde{E} - \tilde{K} \right)^{-1} \tilde{L}, \]
and the system
\[
\begin{align*}
\tilde{E} \dot{w}(t) &= \tilde{K} w(t) + \tilde{L} u(t) \\
\v(t) &= \tilde{M} w(t)
\end{align*}
\]
is positive and stable.

3 Result

The main result of this paper is given in the following theorem.

**Theorem 2** If \( \tilde{N}, \tilde{L}_2, I_{n-n_1}, \tilde{M}_2 \) be a realization of polynomial transfer matrix \( T_{pl}(s) \in \mathbb{R}^{p \times m}(s) \), where \( \tilde{L}_2 \in \mathbb{R}^{(n-n_1) \times m} \), \( \tilde{M}_2 \in \mathbb{R}^{p \times (n-n_1)} \), \( \tilde{N} \in \mathbb{R}^{(n-n_1) \times (n-n_1)} \), with \( \tilde{N}^{q-1} \neq O, \tilde{N}^q = O \), then there exists a positive stable realization of the improper transfer matrix \( T(s) \in \mathbb{R}^{p \times m}(s) \) of the form
\[
\tilde{E} =
\begin{bmatrix}
I_{n_1} & O \\
O & N
\end{bmatrix},
\tilde{K} =
\begin{bmatrix}
\tilde{K}_1 & O \\
O & I_{n-n_1}
\end{bmatrix},
\tilde{L} =
\begin{bmatrix}
\tilde{L}_1 \\
\tilde{L}_2
\end{bmatrix},
\tilde{M} =
\begin{bmatrix}
\tilde{M}_1 & \tilde{M}_2
\end{bmatrix}
\]
if and only if \( \tilde{K}_1, \tilde{L}_1, \tilde{M}_1 \) be a positive stable realization of strictly proper transfer matrix \( T_{sp}(s) \in \mathbb{R}^{p \times m}(s) \).
Proof. Suppose that there exists a positive stable realization of the improper transfer matrix $T(s) \in \mathbb{R}^{p \times m}(s)$ of the form (12), then the system (11) is positive and stable, or equivalently the both systems

$$\begin{align*}
\dot{w}_1(t) &= \bar{K}_1 w_1(t) + \bar{L}_1 u(t) \\
v_1(t) &= \bar{M}_1 w_1(t)
\end{align*}$$

(13)

and

$$\begin{align*}
\bar{N}\dot{w}_2(t) &= w_2(t) + \bar{L}_2 u(t) \\
v_2(t) &= \bar{M}_2 w_2(t),
\end{align*}$$

(14)

where $w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$, are positive and stable. In accordance to Theorem 1 we have $\bar{K}_1 \in \mathcal{M}^{n_1}$, $\bar{L}_1 \in \mathbb{R}^{n_1 \times m}$, $\bar{M}_1 \in \mathbb{R}^{p \times n_1}$. Therefore, these matrices are a positive stable realization of strictly proper transfer matrix $T_{sp}(s) \in \mathbb{R}^{p \times m}(s)$. Otherwise, let $\bar{K}_1, \bar{L}_1, \bar{M}_1$ be a positive stable realization of strictly proper transfer matrix $T_{sp}(s) \in \mathbb{R}^{p \times m}(s)$, then the system (13) is stable positive. Furthermore, since $\bar{N}, \bar{L}_2, I_{n-n_1}, \bar{M}_2$ be a realization of polynomial transfer matrix $T_{pl}(s)$, then

$$T_{pl}(s) = \bar{M}_2 \left( s\bar{N} - I_{n-n_1} \right)^{-1} \bar{L}_2.$$  

We will show that the matrices $\bar{E}, \bar{K}, \bar{L}, \bar{M}$ are a positive stable realization of the improper transfer matrix $T(s)$. Observe that

$$\begin{align*}
T_{sp}(s) + T_{pl}(s) &= \bar{M}_1 \left( sI_{n_1} - \bar{K}_1 \right)^{-1} \bar{L}_1 + \bar{M}_2 \left( s\bar{N} - I_{n_2} \right)^{-1} \bar{L}_2 \\
&= \begin{bmatrix} \bar{M}_1 & \bar{M}_2 \end{bmatrix} \begin{bmatrix} sI_{n_1} - \bar{K}_1 & O \\ O & s\bar{N} - I_{n-n_1} \end{bmatrix}^{-1} \begin{bmatrix} \bar{L}_1 \\ \bar{L}_2 \end{bmatrix} \\
&= \begin{bmatrix} \bar{M}_1 & \bar{M}_2 \end{bmatrix} \left( s \begin{bmatrix} I_{n_1} & O \\ O & \bar{N} \end{bmatrix} - \begin{bmatrix} \bar{K}_1 & O \\ O & I_{n-n_1} \end{bmatrix} \right) \begin{bmatrix} \bar{L}_1 \\ \bar{L}_2 \end{bmatrix} \\
&= \bar{M} \left( s\bar{E} - \bar{K} \right)^{-1} \bar{L} \\
&= T(s).
\end{align*}$$

This shows that $\bar{E}, \bar{K}, \bar{L}, \bar{M}$ be a realization of transfer matrix $T(s)$. The solution of (14) is given by

$$w_2(t) = - \sum_{j=1}^{q-1} \bar{N}^j \bar{L}_2 u^{(j)}(t).$$

(15)

It is obvious that $w_2(t) \in \mathbb{R}^{n-n_1}$ due to $\bar{N}^j \bar{L}_2 \in \mathbb{R}^{(n-n_1) \times m}$, and $w_2(t) \to 0$ when $t \to \infty$. Therefore $\bar{E}, \bar{K}, \bar{L}, \bar{M}$ is a positive stable realization for $T(s)$. ■
4 Conclusion

A criterion for existence of a positive stable realization of an improper transfer matrix for continuous linear singular systems has been established. It is showed that the existence of a positive stable realization of an improper transfer matrix is equivalent to existence of positive stable realization of a strictly proper transfer matrix that corresponding to it.

ACKNOWLEDGEMENTS. This work was supported by Institute for Research and Community Development, Andalas University, Indonesia through Grant No. 003/H.16/PL/HP-MT/III/2010.

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Received: February 10, 2014