Some Remarks about
Almost Semi-hyponormal Operators

Vasile Lauric

Department of Mathematics
Florida A&M University
Tallahassee, FL 32307, USA

Copyright © 2014 Vasile Lauric. This is an open access article distributed under the
Creative Commons Attribution License, which permits unrestricted use, distribution, and
reproduction in any medium, provided the original work is properly cited.

This note is written with the occasion of the 70th birthday of my mother.

Abstract

We define the class of almost semi-hyponormal operators on a Hilbert
space and provide some sufficient conditions in which such operators are
almost normal, that is their self-commutator is in the trace-class.

Mathematics Subject Classification: 47B20

Keywords: essential spectrum, almost normal operators, almost semi-
hyponormal operators

1 Introduction

Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space, and let
$L(\mathcal{H})$ denote the algebra of all linear bounded operators on $\mathcal{H}$, let $\mathbb{K}(\mathcal{H})$ denote
the two-sided ideal of all compact operators on $\mathcal{H}$, and for $p > 0$ let $C_p(\mathcal{H})$
denote the Schatten-von Neumann classes. Although for $0 < p < 1$, the usual
expression of $\| \cdot \|_p$ does not satisfy the triangle inequality, and thus it is only
a quasi-norm, the $C_p(\mathcal{H})$ class is a complete space with respect to $\| \cdot \|_p$.

For $\alpha > 0$ and $T \in L(\mathcal{H})$, we will denote $D_T^\alpha := |T|^{2\alpha} - |T^*|^{2\alpha}$, where
$|T| = (T^*T)^{1/2}$. In particular, $D_T^1 = [T^*, T] = T^*T - TT^* = |T|^2 - |T^*|^2$ =
\[ |T|^2 - U|T|^2 U^* \text{ and } D_T^{\frac{1}{2}} = |T| - |T^*| = |T| - U|T|U^*, \text{ where } U|T| \text{ is the polar decomposition of } T. \]

For \( \alpha, \rho > 0 \), let

\[ N_\rho^\alpha(\mathcal{H}) := \{ N \in L(\mathcal{H}) \mid D_N^\alpha \in \mathcal{C}_\rho(\mathcal{H}) \} \]

and

\[ H_\rho^\alpha(\mathcal{H}) := \{ T \in L(\mathcal{H}) \mid D_T^\alpha = P - K \text{ with } P, K \geq 0, \text{ and } K \in \mathcal{C}_\rho(\mathcal{H}) \}. \]

When \( \alpha = \rho = 1 \), the operators in \( H_1^1(\mathcal{H}), (N_1^1(\mathcal{H}), \text{ respectively) will be called } \text{almost hyponormal} \), (almost normal, respectively), and the operators in \( H_1^{\frac{3}{2}}(\mathcal{H}) \) will be called \text{almost semi-hyponormal} operators. The class \( H_\rho^\alpha(\mathcal{H}) \) can be alternatively be defined (see [6]) as \( \{ T \in L(\mathcal{H}) \mid (D_T^\alpha)_{\text{p}} \in \mathcal{C}_\rho(\mathcal{H}) \} \), where for a self-adjoint operator \( A \in L(\mathcal{H}) \), \( A_{\text{p}} \) denotes its negative part \( (|A| - A)/2 \).

It is easy to verify \( N_1^{\frac{3}{2}}(\mathcal{H}) \subseteq N_1^1(\mathcal{H}) \), but not a similar inclusion concerning \( H_1^{\frac{3}{2}}(\mathcal{H}) \) and \( H_1^1(\mathcal{H}) \).

Planar Lebesgue measure on the \( \sigma \)-ring of Borel sets in the complex plane will be denoted by \( \mu \). The \text{rational cyclic multiplicity} of an operator \( T \) in \( L(\mathcal{H}) \), denoted by \( m(T) \), is the smallest cardinal number \( m \) with the property that there are \( m \) vectors \( x_1, \ldots, x_m \) in \( \mathcal{H} \) such that

\[ \forall \{f(T)x_j \mid 1 \leq j \leq m, \; f \in \mathcal{R}(\sigma(T)) \} = \mathcal{H}, \]

where \( \mathcal{R}(\sigma(T)) \) denotes the algebra of complex-valued rational functions with poles off \( \sigma(T) \), where \( \sigma(T) \) denotes the spectrum of \( T \). D. Voiculescu [9] extended Berger-Shaw Inequality [1] and provided an elegant proof which was dependent only on operator-theoretic concepts. Precisely, Voiculescu proved that if \( T \in H_1^1(\mathcal{H}) \) and there exists \( K \in \mathcal{C}_2(\mathcal{H}) \) so that \( m(T + K) < \infty \), then \( T \in N_1^1(\mathcal{H}) \) and \( \pi \text{tr}(D_T^1) \leq m(T + K) \cdot \mu(\sigma(T + K)) \). The original Berger-Shaw inequality is the special case of this result when \( T \) is hyponormal and \( K = 0 \). In [3], Voiculescu’s result was extended to the case in which \( m(T + K) = \infty \) and \( \mu(\sigma(T + K)) = 0 \), and concluding that \( \text{tr}(D_T^1) \leq 0 \), and consequently, by replacing \( T \) with \( T^* \), that \( \text{tr}(D_T^1) = 0 \). Furthermore, [3] provided necessary and sufficient conditions for almost hyponormal operators with Weyl spectrum of area zero to be almost normal operators, and in addition \( \text{tr}(D_T^1) \) can be expressed in terms of the data of the spectral picture of \( T \). Let \( \mathcal{F}(\mathcal{H}) \) denote the set of all Fredholm operators on \( \mathcal{H} \), and let

\[ \Omega_0(T) := \{ \lambda \in \mathbb{C} \mid T - \lambda \in \mathcal{F}(\mathcal{H}), \; \text{ind} \; (T - \lambda) = 0 \}, \]

and recall that the Weyl spectrum of an operator \( T \in L(\mathcal{H}) \), denoted by \( \sigma_w(T) \), is \( \sigma(T) \setminus \Omega_0(T) \), or alternatively, the union of the essential spectrum of
$T$, $\sigma_e(T)$, and all bounded components of $\mathbb{C} \setminus \sigma_e(T)$ associated with nonzero Fredholm index.

The goal of this note is to prove that the hypotheses $\mu(\sigma_w(T)) = 0$ for almost semi-hyponormal operators $T$, that is $T \in H^\frac{1}{2}_1(\mathcal{H})$, implies $T \in N^\frac{1}{2}_1(\mathcal{H})$, and consequently $T \in N^1_1(\mathcal{H})$.

## 2 Main Results

For $T \in L(\mathcal{H})$, let $\tilde{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2}$ be the Aluthge transform of operator $T$.

**Proposition 2.1.** If $T \in H^\frac{1}{2}_1(\mathcal{H})$, then $\tilde{T} \in H^1_1(\mathcal{H})$.

**Proof.** Let $T \in H^\frac{1}{2}_1(\mathcal{H})$, and write $|T| - |T*| = P_1 - K_1$ with $P_1, K_1 \geq 0$ and $K_1 \in \mathcal{C}_1(\mathcal{H})$. Since $U$ is the partial isometry from $\text{Ran}(|T|)$ onto $\text{Ran}(T)$, $U^*U$ is the orthogonal projection onto $\text{Ran}(|T|)$ and thus $|T*| = U|T|U^*$. Therefore

$$|T| - U|T|U^* = P_1 - K_1,$$

and by multiplying at right side by $U$ and left side by $U^*$,

$$U^*|T|U - |T| = P_2 - K_2,$$

with $P_2, K_2 \geq 0$ and $K_2 \in \mathcal{C}_1(\mathcal{H})$. Multiplying equalities (1) and (2) at both sides by $|T|^\frac{1}{2}$,

$$|T|^2 - ||T|^\frac{1}{2}U|T|^\frac{1}{2}|^2 = P_3 - K_3,$$

and

$$||T|^\frac{1}{2}U|T|^\frac{1}{2}|^2 - |T|^2 = P_4 - K_4,$$

with $P_3, P_4, K_3, K_4 \geq 0$ and $K_3, K_4 \in \mathcal{C}_1(\mathcal{H})$. Adding (3) and (4) leads to

$$|\tilde{T}|^2 - |(\tilde{T})^*|^2 = (P_3 + P_4) - (K_3 + K_4),$$

that is the desired conclusion. \hfill $\square$

Stampfli [7] proved that there exists $K \in \mathbb{K}(\mathcal{H})$ such that $\sigma(T + K) = \sigma_w(T)$. A careful review of Stampfli’s proof leads to the following statement that will be necessary later.

**Lemma 2.2.** ([7]). Let $T \in L(\mathcal{H})$ and $p \geq 0$. Then for any $\varepsilon > 0$, there exists $K \in \mathcal{C}_p(\mathcal{H})$ such that $||K||_p < \varepsilon$ and $\sigma(T + K) \setminus \sigma_w(T)$ consists of a countable set which clusters only on $\sigma_w(T)$.

Let $Q_0(\mathcal{H})$ denote $\{T \in L(\mathcal{H}) \mid 0 \in \rho_{le}(T) \cup \rho_{re}(T)\}$, where $\rho_{le}(T)$, $\rho_{re}(T)$ are the left essential and right essential resolvent of the operator $T \in L(\mathcal{H})$, respectively. Equivalently, $Q_0(\mathcal{H})$ is $\{T \in L(\mathcal{H}) \mid T^*T$ or $TT^*$ belongs to $\mathcal{F}(\mathcal{H})\}$, or $\{T \in L(\mathcal{H}) \mid |T|$ or $|T^*|$ belongs to $\mathcal{F}(\mathcal{H})\}$. 
Theorem 2.3. Let \( T \in H^\frac{3}{2}(\mathcal{H}) \cap Q_0(\mathcal{H}) \) such that \( \mu(\sigma_w(T)) = 0. \) Then \( T \in N^\frac{3}{2}(\mathcal{H}). \)

Proof. According to [4-5], \( \sigma(T) = \sigma(\tilde{T}) \sigma_e(T) = \sigma_e(\tilde{T}) \) and for any \( \lambda \in \mathbb{C} \), then \( T - \lambda \in \mathcal{F}(\mathcal{H}) \) if and only if \( \tilde{T} - \lambda \in \mathcal{F}(\mathcal{H}) \) and \( \text{ind}(T - \lambda) = \text{ind}(\tilde{T} - \lambda). \) Thus \( \sigma_w(T) = \sigma_w(\tilde{T}) \), and thus \( \mu(\sigma_w(\tilde{T})) = 0 \) for an operator \( T \) as in the hypothesis. According to Lema 2.2, there exists \( K \in \mathcal{C}_1(\mathcal{H}) \) so that \( \sigma(\tilde{T} + K) \setminus \sigma_w(\tilde{T}) \) is a countable set, and thus \( \mu(\sigma(\tilde{T} + K)) = 0. \) According to Proposition 2.1, \( \tilde{T} \in H^1_1(\mathcal{H}) \), and according to [3], \( \tilde{T} \in N^1_1(\mathcal{H}) \) and \( \text{tr}([\tilde{T}^*, \tilde{T}]) = 0. \) This implies that \( P_3 = |T|^\frac{1}{2}P_1 |T|^\frac{1}{2} \) and \( P_4 = |T|^\frac{1}{2}U^* P_1 U |T|^\frac{1}{2} \), are in \( \mathcal{C}_1(\mathcal{H}), \) \( (P_1, P_3, P_4 \) are the ones that show in the proof of Proposition 2.1). Furthermore, \( P_3 = |T|^\frac{1}{2}P_1 |T|^\frac{1}{2} \in \mathcal{C}_1(\mathcal{H}) \) is equivalent to \( |T|P_1, P_1 |T| \in \mathcal{C}_1(\mathcal{H}), \) and similarly, \( P_4 = |T|^\frac{1}{2}U^* P_1 U |T|^\frac{1}{2} \in \mathcal{C}_1(\mathcal{H}) \) is equivalent to \( |T^*|P_1, P_1 |T^*| \in \mathcal{C}_1(\mathcal{H}). \)

If \( T \in Q_0(\mathcal{H}), \) then \( P_1 \) must be a trace-class operator, that is \( T \in N^\frac{3}{2}(\mathcal{H}). \) \( \square \)

Hadwin-Nordgren [3] gave a necessary and sufficient characterization for almost hyponormal operators with essential spectrum of area zero. For \( T \in L(\mathcal{H}), \) let \( V_1, V_2, \ldots \) be the bounded components of \( \mathbb{C} \setminus \sigma_e(T) \) and \( m_1, m_2, \ldots \) be the associated Fredholm indices. Using Voiculescu’s idea of “filling the bounded holes” of \( \sigma_e(T) \) with direct sums of \( |m_k| \) copies of some translations the unilateral shift or its adjoint, they showed the following.

With the above notation, the following two statements hold.

Theorem 2.4. ([3]). Let \( T \in H^1_1(\mathcal{H}) \) such that \( \mu(\sigma_e(T)) = 0 \) and

\[
\sum_{m_k < 0} m_k \cdot \mu(V_k) > -\infty.
\]

Then \( T \in N^1_1(\mathcal{H}), \sum_k |m_k| \cdot \mu(V_k) < +\infty, \) and

\[
\pi \cdot \text{tr}([T^*, T]) = -\sum_k m_k \cdot \mu(V_k).
\]

Consequently, an almost hyponormal operator \( T \) with \( \mu(\sigma_e(T)) = 0 \) is almost normal if and only if \( \sum_{m_k < 0} m_k \cdot \mu(V_k) > -\infty. \)

Theorem 2.5. Let \( T \in H^\frac{3}{2}(\mathcal{H}) \cap Q_0(\mathcal{H}) \) such that \( \mu(\sigma_e(T)) = 0 \) and

\[
\sum_{m_k < 0} m_k \cdot \mu(V_k) > -\infty.
\]

Then \( T \in N^\frac{3}{2}(\mathcal{H}). \) Consequently \( [T^*, T] \in \mathcal{C}_1(\mathcal{H}), \sum_k |m_k| \cdot \mu(V_k) < +\infty, \) and \( \pi \cdot \text{tr}([T^*, T]) = -\sum_k m_k \cdot \mu(V_k). \)

Furthermore, an almost semi-hyponormal operator \( T \) with \( \mu(\sigma_e(T)) = 0 \) is almost normal if and only if \( \sum_{m_k < 0} m_k \cdot \mu(V_k) > -\infty. \)
Some remarks about almost semi-hyponormal operators

Proof. Write $|T| - |T^*| = P_1 - K_1$, with $P_1, K_1 \geq 0$ and $K_1 \in C_1(\mathcal{H})$. According to Proposition 2.1, $\tilde{T} \in H'_1(\mathcal{H})$. Again, according to [4-5], the spectral pictures of $T$ and $\tilde{T}$ are identical, that is the bounded components of $\mathbb{C} \setminus \sigma_e(\tilde{T})$, and their associated Fredholm indices are the same as of $\mathbb{C} \setminus \sigma_e(T)$, and according to Theorem 2.4, $\tilde{T} \in N_1^1(\mathcal{H})$. Using the same argument as in the proof of Theorem 2.3, operators $P_3, P_4$ that appear in proof of Proposition 2.1 are trace-class, and when $T \in Q_0(\mathcal{H})$, the operator $P_1$ is also trace-class, and thus $T \in N_{1}^{2}(\mathcal{H}) \subseteq N_{1}^{1}(\mathcal{H})$. The other conclusions of the theorem are routine consequences of Theorem 2.4.

An obvious consequence of Brown-Douglas-Fillmore Theorem [2] is the following.

Corollary 2.6. If $S, T \in H_{1}^{\frac{1}{2}}(\mathcal{H}) \cap Q_{0}(\mathcal{H})$ such that $\mu(\sigma_e(T)) = 0$ and if there exists a unitary operator $U$ such that $UTU^* - S \in \mathcal{K}(\mathcal{H})$, then $T \in H_{1}^{1}(\mathcal{H})$ iff $S \in H_{1}^{1}(\mathcal{H})$ and $tr[T^*, T] = tr[S^*, S]$.

Remarks 2.7. (a) It would be interesting to know whether the hypothesis $m(T) < \infty$ implies $m(\tilde{T}) < \infty$. If such a statement were true, then the circle of ideas seen above can be used to prove that $T \in H_{1}^{\frac{1}{2}}(\mathcal{H}) \cap Q_{0}(\mathcal{H})$ with $m(T) < \infty$ implies $T \in N_{1}^{2}(\mathcal{H})$.

(b) Uchiyama [8] proved that if $T$ is $p$-hyponormal and $m(T) < \infty$, then $m(T) = m(T)$. In fact the same consequence can be obtained for operators $T \in L(\mathcal{H})$ such that $\lceil \text{Ran}(T) \rceil \subseteq \lceil \text{Ran}(T^*) \rceil$.

References


Received: January 30, 2014