Higher-Order Daehee Numbers and Polynomials

Dae San Kim
Department of Mathematics, Sogang University
Seoul 121-742, Republic of Korea

Taekyun Kim
Department of Mathematics, Kwangwoon University
Seoul 139-701, Republic of Korea

Sang-Hun Lee
Division of General Education, Kwangwoon University
Seoul 139-701, Republic of Korea

Jong-Jin Seo
Department of Applied Mathematics
Pukyong National University
Pusan, Republic of Korea

Copyright © 2014 Dae San Kim, Taekyun Kim, Sang-Hun Lee and Jong-Jin Seo. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

Recently, Daehee numbers and polynomials are introduced by the authors. In this paper, we consider the Daehee numbers and polynomials of order $k \in \mathbb{N}$ and give some relation between Daehee polynomials of order $k \in \mathbb{N}$ and special polynomials.
1. Introduction

For $\alpha \in \mathbb{N}$, as is well known, the Bernoulli polynomials of order $\alpha$ are defined by the generating function to be

$$\left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)} (x) \frac{t^n}{n!},$$

(1)

(see [1-14]).

When $x = 0$, $B_n^{(\alpha)} = B_n^{(\alpha)} (0)$ are the Bernoulli numbers of order $\alpha$. In [4, 7, 12], the Daehee polynomials are defined by the generating function to be

$$\left( \log (1 + t) \right) \frac{(1 + t)^x}{t} = \sum_{n=0}^{\infty} D_n (x) \frac{t^n}{n!}.$$

(2)

When $x = 0$, $D_n = D_n (0)$ are called the Daehee numbers.

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm $| \cdot |_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $\text{UD} (\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \text{UD} (\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I (f) = \int_{\mathbb{Z}_p} f (x) \, d\mu (x) = \lim_{n \to \infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f (x),$$

(3)

(see [10]).

Let $f_1 (x) = f (x + 1)$. Then, by (3), we get

$$I (f_1) - I (f) = f' (0), \text{ where } f' (0) = \left. \frac{df (x)}{dx} \right|_{x=0}.$$

(4)

The signed Stirling numbers of the first kind $S_1 (n, l)$ are defined by

$$(x)_n = x (x - 1) \cdots (x - n + 1) = \sum_{l=0}^{\infty} S_1 (n, l) x^l,$$

(5)

(see [2, 3, 4]).

From (5), we note that

$$x^{(n)} = x (x + 1) \cdots (x + n - 1) = (-1)^n (-x)_n$$

$$= \sum_{l=0}^{n} (-1)^{n-l} S_1 (n, l) x^l,$$

(see [4, 5, 12]).
Higher-Order Daehee numbers and polynomials

The Stirling numbers of the second kind $S_2(l, n)$ are defined by the generating function to be

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}$$

$$= \sum_{l=0}^{\infty} \frac{n!}{(l+n)!} S_2(l + n, n) t^{l+n}.$$  

In this paper, we study the higher-order Daehee numbers and polynomials and give some relations between Daehee polynomials and special polynomials.

2. Higher-order Daehee polynomials

In this section, we assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{\frac{1}{p-1}}$. For $k \in \mathbb{N}$, let us consider the Daehee numbers of the first kind of order $k$ :

$$D_n^{(k)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + x_2 + \cdots + x_k)_n d\mu(x_1) \cdots d\mu(x_k),$$

where $n \in \mathbb{Z}_{\geq 0}$.

From (7), we can derive the generating function of $D_n^{(k)}$ as follows :

$$\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \left( \frac{x_1 + \cdots + x_k}{n} \right) t^n d\mu(x_1) \cdots d\mu(x_k)$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + t)^{x_1 + \cdots + x_k} d\mu(x_1) \cdots d\mu(x_k).$$

By (4), we easily see that

$$\int_{\mathbb{Z}_p} (1 + t)^2 d\mu(x) = \frac{\log (1 + t)}{t}.$$  

Thus, by (8) and (9), we get

$$\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} = \left( \frac{\log (1 + t)}{t} \right)^k.$$  

Now, we observe that
\[
\left( \frac{\log (1 + t)}{t} \right)^k = \frac{k!}{t^k} \sum_{l=k}^{\infty} S_1(t, k) \frac{t^l}{l!} = \sum_{n=0}^{\infty} S_1(n + k, k) \frac{k!}{(n + k)!} t^n = \sum_{n=0}^{\infty} S_1(n + k, k) \frac{t^n}{(n + k)!}.
\] (11)

Therefore, by (10) and (11), we obtain the following theorem.

**Theorem 1.** For \( n \in \mathbb{Z}_{\geq 0}, \ k \in \mathbb{N}, \) we have
\[
D_n^{(k)} = \frac{S_1(n + k, k)}{{n + k \choose k}}.
\]

It is easy to show that
\[
\left( \frac{\log (1 + t)}{t} \right)^k = \sum_{n=0}^{\infty} B^{(n+k+1)}_n(1) \frac{t^n}{n!}.
\] (12)

Therefore, we obtain the following corollary.

**Corollary 2.** For \( n \in \mathbb{Z}_{\geq 0}, \ k \in \mathbb{N}, \) we have
\[
D_n^{(k)} = \frac{S_1(n + k, k)}{{n + k \choose k}} = B^{(n+k+1)}_n(1).
\]

From (7), we note that
\[
D_n^{(k)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k)^n d\mu(x_1) \cdots d\mu(x_k)
\] (13)
\[
= \sum_{l=0}^n S_1(n, l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k)^l d\mu(x_1) \cdots d\mu(x_k)
\]
\[
= \sum_{l=0}^n S_1(n, l) B_l^{(k)}.
\]

Therefore, by (13), we obtain the following theorem.

**Theorem 3.** For \( n \in \mathbb{Z}_{\geq 0}, \ k \in \mathbb{N}, \) we have
\[
D_n^{(k)} = \sum_{l_1 + \cdots + l_k = n} \binom{n}{l_1, \ldots, l_k} D_{l_1} \cdots D_{l_k}
\]
\[
= \sum_{l=0}^n S_1(n, l) B_l^{(k)}.
\]
From (10), we can derive

\[ \sum_{n=0}^{\infty} D_n^{(k)} \frac{(e^t - 1)^n}{n!} = \left( \frac{t}{e^t - 1} \right)^k = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \]  

(14)

and

\[ \sum_{n=0}^{\infty} D_n^{(k)} \frac{(e^t - 1)^n}{n!} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} D_n^{(k)} S_2(n, m) \right) \frac{t^n}{n!}. \]  

(15)

Therefore, by (14) and (15), we obtain the following theorem.

**Theorem 4.** For \( m \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N}, \) we have

\[ B_m^{(k)} = \sum_{n=0}^{m} D_n^{(k)} S_2(n, m). \]

Now, we consider the higher-order Daehee polynomials as follows:

\[ D_n^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)^n d\mu(x_1) \cdots d\mu(x_k). \]  

(16)

Thus, by (16), we get

\[ D_n^{(k)}(x) = \sum_{l=0}^{n} S_1(n, l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)^l d\mu(x_1) \cdots d\mu(x_k) \]

\[ = \sum_{l=0}^{n} S_1(n, l) B_l^{(k)}(x). \]

Therefore, by (17), we obtain the following theorem.

**Theorem 5.** For \( n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N}, \) we have

\[ D_n^{(k)}(x) = \sum_{l=0}^{n} S_1(n, l) B_l^{(k)}(x). \]

From (16), we derive the generating function of \( D_n^{(k)}(x): \)
\[
\sum_{n=0}^{\infty} D^{(k)}_n (x) \frac{t^n}{n!}
\]
\[
= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} (x_1 + \cdots + x_k + x) t^n d\mu (x_1) \cdots d\mu (x_k)
\]
\[
= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + t)^{x_1 + \cdots + x_k + x} d\mu (x_1) \cdots d\mu (x_k)
\]
\[
= \left( \frac{\log (1 + t)}{t} \right)^k (1 + t)^x.
\]

It is easy to show that
\[
\left( \frac{\log (1 + t)}{t} \right)^k (1 + t)^x = \sum_{n=0}^{\infty} B^{(n+k+1)}_n (x + 1) \frac{t^n}{n!}.
\]

Therefore, by (18) and (19), we obtain the following theorem.

**Theorem 6.** For \( n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N} \),
\[
D^{(k)}_n (x) = B^{(n+k+1)}_n (x + 1)
\]
\[
= \sum_{l=0}^{n} \binom{n}{l} B^{(n+k+1)}_l (x + 1)^{n-l}.
\]

In (18), we note that
\[
\sum_{n=0}^{\infty} D^{(k)}_n (x) \frac{(e^t - 1)^n}{n!} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} S_2 (n, m) D^{(k)}_n (x) \right) \frac{t^m}{m!}
\]
and
\[
\sum_{n=0}^{\infty} D^{(k)}_n (x) \frac{(e^t - 1)^n}{n!} = \left( \frac{t}{e^t - 1} \right)^k e^{xt}
\]
\[
= \sum_{m=0}^{\infty} B^{(k)}_m (x) \frac{t^m}{m!}.
\]

Therefore, by (20) and (21), we obtain the following theorem.

**Theorem 7.** For \( m \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N} \), we have
\[
B^{(k)}_m (x) = \sum_{n=0}^{m} S_2 (m, n) D^{(k)}_n (x).
\]
Now, we define Daehee numbers of the second kind of order \(k\) \((k \in \mathbb{N})\):

\[
\hat{D}_n^k[(k)] = (-1)^n \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} \left(-x_1 - x_2 - \cdots - x_k\right)_n d\mu(x_1) \cdots d\mu(x_k)
\]

\[
= (-1)^n \sum_{l=0}^{n} (-1)^{n-l} S_1(n, l) B_{l}^{(k)} = \sum_{l=0}^{n} \left[\begin{array}{c} n \\ l \end{array}\right] B_{l}^{(k)},
\]

where \( \left[\begin{array}{c} n \\ l \end{array}\right] = (-1)^{n-l} S_1(n, l) \).

Thus, by (22), we get

\[
\hat{D}_n^k[(k)] = (-1)^n \sum_{l=0}^{n} (-1)^{n-l} S_1(n, l) \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} \left(1 - (x_1 + x_2 + \cdots + x_k)\right)_l d\mu(x_1) \cdots d\mu(x_k)
\]

\[
= \sum_{l=0}^{n} (-1)^{n-l} S_1(n, l) B_{l}^{(k)} = \sum_{l=0}^{n} \left[\begin{array}{c} n \\ l \end{array}\right] B_{l}^{(k)},
\]

where \( \left[\begin{array}{c} n \\ l \end{array}\right] = (-1)^{n-l} S_1(n, l) \).

Therefore, by (23), we obtain the following theorem.

**Theorem 8.** For \(n \in \mathbb{Z}_{>0}, k \in \mathbb{N}\), we have

\[
\hat{D}_n^k[(k)] = \sum_{l=0}^{n} \left[\begin{array}{c} n \\ l \end{array}\right] B_{l}^{(k)}.
\]

From (22), we derive the generating function of \(\hat{D}_n^k[(k)]\):

\[
\sum_{n=0}^{\infty} \hat{D}_n^k[(k)] \frac{t^n}{n!} = \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} \sum_{n=0}^{\infty} \left(x_1 + \cdots + x_k + n - 1\right)_n \sum_{n=0}^{\infty} \left(1 - t\right)^{-x_1 - \cdots - x_k} d\mu(x_1) \cdots d\mu(x_k)
\]

\[
= \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} \left(1 - t\right)^{-x_1 - \cdots - x_k} d\mu(x_1) \cdots d\mu(x_k)
\]

\[
= \left(\frac{(1 - t) \log (1 - t)}{-t}\right)^k.
\]
By (24), we get
\[ \sum_{n=0}^{\infty} \hat{D}^k_n[(k)] \frac{(1 - e^{-t})^n}{n!} = \left( \frac{e^{-t}(-t)}{e^t - 1} \right)^k = \left( \frac{t}{e^t - 1} \right)^k \]
\[ = \sum_{m=0}^{\infty} B_m^{(k)} \frac{t^m}{m!}, \tag{25} \]
and
\[ \sum_{n=0}^{\infty} \hat{D}^k_n[(k)] \frac{(1 - e^{-t})^n}{n!} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \hat{D}^k_n[(k)] (-1)^{m-n} S_2(m,n) \right) \frac{t^m}{m!}. \tag{26} \]

Therefore, by (25) and (26), we obtain the following theorem.

**Theorem 9.** For \( m \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N} \), we have
\[ B_m^{(k)} = \sum_{n=0}^{m} \hat{D}^k_n[(k)] (-1)^{n-m} S_2(m,n). \]

Now, we consider the higher-order Dahee polynomials of the second kind:
\[ \hat{D}^k_n[(k)](x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + x_2 + \cdots + x_k - x)^{(n)} d\mu(x_1) \cdots d\mu(x_k). \tag{27} \]

Thus, by (27), we get
\[ \hat{D}^k_n[(k)](x) = (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - x_2 - \cdots - x_k + x)^{(n)} d\mu(x_1) \cdots d\mu(x_k) \]
\[ = (-1)^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - x_2 - \cdots - x_k + x)^l d\mu(x_1) \cdots d\mu(x_k) \]
\[ = (-1)^n \sum_{l=0}^{n} S_1(n,l) \sum_{m=0}^{l} \binom{l}{m} x^{l-m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - x_2 - \cdots - x_k)^m d\mu(x_1) \cdots d\mu(x_k) \]
\[ = (-1)^n \sum_{l=0}^{n} S_1(n,l) \sum_{m=0}^{l} \binom{l}{m} (-1)^m x^{l-m} B_m^{(k)} \]
\[ = \sum_{l=0}^{n} (-1)^{n-l} S_1(n,l) B_l^{(k)}(-x). \]

Thus, by (28), we get
\[ \hat{D}^k_n[(k)](x) = \sum_{l=0}^{n} (-1)^{n-l} S_1(n,l) B_l^{(k)}(-x). \tag{29} \]
Let us consider the generating function of $D_n^{(k)}(x)$ as follows:

$$
\sum_{n=0}^{\infty} \hat{D}_n^{(k)}(k)(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \frac{x_1 + \cdots + x_k - x + n - 1}{n} t^n d\mu(x_1) \cdots d\mu(x_k)
$$

$$
= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1-t)^{-x_1-\cdots-x_k+x} d\mu(x_1) \cdots d\mu(x_k)
$$

$$
= \left( \frac{(1-t) \log (1-t)}{-t} \right)^k (1-t)^x.
$$

From (30), we have

$$
\sum_{n=0}^{\infty} \hat{D}_n^{(k)}(k)(x) (-1)^n \frac{t^n}{n!}
$$

$$
= \left( \frac{\log (1+t)}{t} \right)^k (1+t)^{x+k}
$$

$$
= \sum_{n=0}^{\infty} B_{n+k+1}^{(n+k+1)} (x+k+1) \frac{t^n}{n!}.
$$

Therefore, by (31), we obtain the following theorem.

**Theorem 10.** For $n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, we have

$$
(-1)^n \hat{D}_n^{(k)}(k)(x) = B_{n+k+1}^{(n+k+1)} (x+k+1).
$$

By (30), we get

$$
\sum_{n=0}^{\infty} \hat{D}_n^{(k)}(k)(x) \frac{(1-e^{-t})^n}{n!} = e^{-tx} \left( \frac{t}{e^t - 1} \right)^k
$$

$$
= \sum_{m=0}^{\infty} B_{m}^{(k)} (-x) \frac{t^m}{m!},
$$

and

$$
\sum_{n=0}^{\infty} \hat{D}_n^{(k)}(k)(x) \frac{1}{n!} (1-e^{-t})^{-n}
$$

$$
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \hat{D}_n^{(k)}(k)(x) (-1)^{m-n} S_2(m,n) \right) \frac{t^m}{m!}.
$$

Therefore, by (32) and (12), we obtain the following theorem.
Theorem 11. For \( m \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N} \), we have
\[
B_m^{(k)} (-x) = \sum_{n=0}^{m} \hat{D}_n^k [(k)] (x) (-1)^{m-n} S_2 (m, n).
\]

Now, we observe that
\[
(-1)^n \frac{D_n^k (x)}{n!}
= (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{x_1 + \cdots + x_k + x}{n} \right) d\mu (x_1) \cdots d\mu (x_k)
= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( -(x_1 + \cdots + x_k) - x + n - 1 \right) d\mu (x_1) \cdots d\mu (x_k)
= \sum_{m=0}^{n} \left( \frac{n-1}{n-m} \right) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( -(x_1 + \cdots + x_k) - x \right) d\mu (x_1) \cdots d\mu (x_k)
= \sum_{m=0}^{n} \left( \frac{n-1}{n-m} \right) m! \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( -(x_1 + \cdots + x_k) - x \right) d\mu (x_1) \cdots d\mu (x_k)
= \sum_{m=1}^{n} \left( \frac{n-1}{n-m} \right) m! \left( -1 \right)^m \hat{D}_m^k (-x).
\]

Therefore, by (34), we obtain the following theorem.

Theorem 12. For \( n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N} \), we have
\[
(-1)^n \frac{D_n^k (x)}{n!} = \sum_{m=1}^{n} \left( \frac{n-1}{n-m} \right) m! (-1)^m \hat{D}_m^k (-x).
\]

By the same method as Theorem 12, we get
\[
\frac{\hat{D}_n^k [(k)] (x)}{n!}
= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{x_1 + \cdots + x_k - x + n - 1}{n} \right) d\mu (x_1) \cdots d\mu (x_k)
= \sum_{m=0}^{n} \left( \frac{n-1}{n-m} \right) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( x_1 + \cdots + x_k - x \right) d\mu (x_1) \cdots d\mu (x_k)
= \sum_{m=0}^{n} \left( \frac{n-1}{n-m} \right) m! \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( x_1 + \cdots + x_k - x \right) d\mu (x_1) \cdots d\mu (x_k)
= \sum_{m=1}^{n} \left( \frac{n-1}{n-m} \right) m! D_m^k (-x).
\]
Thus, by (35), we get
\[
\hat{D}_k^n[(k)](x) = \sum_{m=1}^{n} \frac{n^{m-1}}{m!} D_m^{(k)}(-x).
\] (36)

REFERENCES


Received: January 5, 2014